# Comparison of Rayleigh's, Herring's, and Gilmore's Models of Gas Bubbles

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Summary

Significant bubble wall positions computed in Rayleigh's, Herring's, modified Herring's, and Gilmore's models of freely oscillating gas bubbles are mutually compared. The computations were performed for a non-linear amplitude of bubble oscillations, $A$, ranging from 1 to 10. $A$ is defined as the ratio of the maximum ($R_M$) to the equilibrium ($R_e$) radius of a bubble. Gilmore's model is used as a reference model and it is found that Rayleigh's model gives satisfactory results for amplitudes $A \leqq 2$, and both the original and the modified Herring models for amplitudes $A \leqq 4.5$.

*Vergleich des Rayleighschen, Herringschen und Gilmoreschen Gasblasenmodells*

Zusammenfassung

Es werden die Lagen einer Blasenwand, berechnet nach dem Rayleighschen, dem Herringschen, einem modifizierten Herringschen und dem Gilmoreschen Modell freischwingender Gasblasen miteinander verglichen. Die Berechnungen werden durchgeführt für einen Wertebereich der nichtlinearen Schwingungsamplitude $A$ zwischen 1 und 10. Dabei ist $A$ als das Verhältnis des Maximalradius zum Gleichgewichtsradius einer Blase definiert. Das Gilmoresche Modell wird als Referenzmodell benutzt; es zeigt sich, daß das Rayleighsche Modell zufriedenstellende Ergebnisse für Amplituden $A = 2$, das ursprüngliche und das modifizierte Herringsche Modell dagegen für Amplituden $A = 4{,}5$ liefert.

*Comparaison des modèles de Rayleigh, de Herring et de Gilmore concernant les bulles de gaz*

Sommaire

On compare entre elles les positions significatives de la paroi d'une bulle de gaz en oscillation libre, ces positions ayant été calculées successivement à partir du modèle de Rayleigh, du modèle d'Herring, du modèle d'Herring modifié et du modèle de Gilmore. Les calculs ont été effectués pour diverses valeurs du paramètre $A$ («amplitude» non-linéaire des oscillations d'une bulle) comprises entre 1 et 10. On définit $A$ comme le rapport entre la valeur maximale $R_M$ et la valeur à l'équilibre $R_e$ du rayon de la bulle. Le modèle de Gilmore a été pris comme élément de référence. La comparaison montre que le modèle de Rayleigh donne satisfaction aux amplitudes $A$ inférieures ou égales à 2 et que les deux modèles d'Herring restent satisfaisants jusqu'à $A = 4{,}5$.

## 1. Introduction

At the present time there are a number of bubble models, such as Rayleigh's [1–4], Herring's [2, 3], Gilmore's [3, 5], Keller's and Kolodner's [6], Flynn's [7], and Tomita's and Shima's [8], that differ both in complexity and in the range of amplitudes for which they can be used. For moderate amplitudes all the models behave essentially in the same way. However, in the region of violent oscillations only the more complex models give satisfactory results.

The bubble models mentioned are formed by a system of equations that enable us to determine the motion of the bubble wall, pressure and temperature fields in the liquid and in the gas, etc. From a mathematical point of view a solution of all these equations by current numerical methods represents no serious problem. However, the more complex models, apart from consuming larger amounts of machine time for computations, are also less "transparent" in providing insight into the processes involved. Hence, it seems only natural always to prefer the simpler models when they are appropriate.

It is the aim of this paper to compare the performance of Rayleigh's, Herring's, and Gilmore's models for different amplitudes and in this way to determine their areas of validity. Besides the three models a modified version of Herring's model, which in spite of its simplicity performs remarkably well, will also be considered. The models of Keller and Kolodner, Flynn, and Tomita and Shima are

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omitted from the comparison because we believe they behave similarly to Herring's model.

A comparison of various bubble models has also been recently reported by Lastman and Wentzell [9]. However, these authors carried out computations only for two values of amplitude ($A = 73.4$, and $A = 2.25$) and for two bubble sizes ($R_e = 10$ μm, and $R_e = 100$ μm) and thus their results do not allow a closer determination of the areas for which the models mentioned above are valid.

In this work the medium-sized bubbles (approximately 3 mm $\leqq R_M \leqq 0.3$ m), for which the effects of gravity, surface tension, viscosity, and heat conduction can be neglected [4], are considered. Computations were carried out for amplitudes $1 < A \leqq 10$ and the bubble was assumed to have a spherical form throughout its motion.

## 2. The bubble models

Here only the equations of motion for the respective models will be given. Further details regarding the models can be found in the sources quoted below.

*Rayleigh's model [1–4]*

In this model the liquid is assumed to be incompressible, which entails an infinite velocity of sound in the liquid. The equation of motion for the bubble wall has the form

$$\ddot{R}R + \tfrac{3}{2}\dot{R}^2 = \frac{1}{\varrho_\infty}(P - p_\infty)\,, \tag{1}$$

where $R$ is the bubble radius, $\varrho_\infty$ is the liquid density at infinity, and finally $P$ and $p_\infty$ are the pressure in the liquid at the bubble wall and at infinity, respectively. The dots denote differentiation with respect to time.

Because of the assumption of liquid incompressibility, this model gives satisfactory results only for small amplitudes of oscillations, when flow velocities are relatively low.

*Herring's original model [2, 3]*

This model is based on the so-called acoustical approximation, which assumes a constant velocity of sound in the liquid, i.e. $c = c_\infty =$ const. The equation of motion has the form

$$\ddot{R}R\left(1 - 2\frac{\dot{R}}{c_\infty}\right) + \tfrac{3}{2}\dot{R}^2\left(1 - \tfrac{4}{3}\frac{\dot{R}}{c_\infty}\right) = \frac{1}{\varrho_\infty}\left[P - p_\infty + \frac{R}{c_\infty}\dot{P}\left(1 - \frac{\dot{R}}{c_\infty}\right)\right]. \tag{2}$$

It can be immediately seen that for $c_\infty \to \infty$ eq. (2) reduces to (1).

This model is suitable for small and medium amplitudes, when the flow velocities are moderate.

*Gilmore's model [3, 5]*

In this model the velocity of sound in the liquid, $c$, varies with the pressure, $p$, as

$$c = c_\infty\left(\frac{p + B}{p_\infty + B}\right)^{\frac{n-1}{2n}}. \tag{3}$$

Here $B$ and $n$ are constants in the Tait equation of state for the liquid. For small deviations of $p$ from $p_\infty$ eq. (3) gives $c \doteqdot c_\infty =$ const.

A further quantity occurring in this model is an enthalpy difference between the liquid at pressures $p$ and $p_\infty$ under isentropic conditions. The enthalpy difference, $h$, equals

$$h = \frac{1}{\varrho_\infty}\frac{n}{n-1}(p_\infty + B)\left[\left(\frac{p + B}{p_\infty + B}\right)^{\frac{n-1}{n}} - 1\right]. \tag{4}$$

For $p, p_\infty \ll B$ eq. (4) can be rewritten in the form $h \doteqdot (p - p_\infty)/\varrho_\infty$.

At the bubble wall $p = P$, $c = C$, and $h = H$. The equation of motion in Gilmore's model then has the form

$$\ddot{R}R\left(1 - \frac{\dot{R}}{C}\right) + \tfrac{3}{2}\dot{R}^2\left(1 - \frac{1}{3}\frac{\dot{R}}{C}\right) = H\left(1 + \frac{\dot{R}}{C}\right) + \frac{R}{C}\dot{H}\left(1 - \frac{\dot{R}}{C}\right). \tag{5}$$

For small deviations of $P$ from $p_\infty$ we obtain from (5) an equation, which differs from (2) only in the form and number of the correction terms $(1 - \dot{R}/c_\infty)$.

Gilmore's model is suitable even for the largest amplitudes of the bubble oscillations.

*Herring's modified model*

As mentioned in Section 1, a modified version of Herring's model will also be considered here. The modification consists in the omission of the correction terms of the form $(1 - \dot{R}/c_\infty)$ from eq. (2). We obtain

$$\ddot{R}R + \tfrac{3}{2}\dot{R}^2 = \frac{1}{\varrho_\infty}\left(P - p_\infty + \frac{R}{c_\infty}\dot{P}\right). \tag{6}$$

This model, like Herring's original model, is suitable for small and medium amplitudes of bubble oscillations.

The *gas pressure* on the walls of the bubble is given by the relation [4, 5]

$$P = P_0\left(\frac{R_0}{R}\right)^{3\gamma}, \tag{7}$$

where $P_0$ and $R_0$ are the initial gas pressure and bubble radius, respectively, and $\gamma$ is the ratio of the principal specific heats.

For further work it is convenient to *normalize* the above equations. In this study we shall use a compression system [4], in which $P_0 = P_m$ and $R_0 = R_M$ (here $R_M$ is the maximum bubble radius and $P_m$ is the gas pressure when $R = R_M$). The non-dimensional compression variables are defined as

$$t_z = t/[R_M (p_\infty/\varrho_\infty)^{1/2}], \quad Z = R/R_M,$$

$$P^* = P/p_\infty, \quad C^* = C(\varrho_\infty/p_\infty)^{1/2}, \quad H^* = H(\varrho_\infty/p_\infty).$$

After normalization we obtain:

*Rayleigh's model*

$$\ddot{Z}Z + \tfrac{3}{2}\dot{Z}^2 = P^* - 1, \tag{8}$$

*Herring's original model*

$$\ddot{Z}Z\left(1 - 2\frac{\dot{Z}}{c_\infty^*}\right) + \tfrac{3}{2}\dot{Z}^2\left(1 - \frac{4}{3}\frac{\dot{Z}}{c_\infty^*}\right) = P^* - 1 + \frac{Z}{c_\infty^*}\dot{P}\left(1 - \frac{\dot{Z}}{c_\infty^*}\right), \tag{9}$$

*Gilmore's model*

$$\ddot{Z}Z\left(1 - \frac{\dot{Z}}{C^*}\right) + \tfrac{3}{2}\dot{Z}^2\left(1 - \frac{1}{3}\frac{\dot{Z}}{C^*}\right) = H^*\left(1 + \frac{\dot{Z}}{C^*}\right) + \frac{Z}{C^*}\dot{H}^*\left(1 - \frac{\dot{Z}}{C^*}\right), \tag{10}$$

$$C^* = c_\infty^* \left(\frac{P^* + B^*}{1 + B^*}\right)^{\frac{n-1}{2n}}, \tag{11}$$

$$H^* = \frac{n}{n-1}(1 + B^*)\left[\left(\frac{P^* + B^*}{1 + B^*}\right)^{\frac{n-1}{n}} - 1\right], \tag{12}$$

*Herring's modified model*

$$\ddot{Z}Z + \tfrac{3}{2}\dot{Z}^2 = P^* - 1 + \frac{Z}{c_\infty^*}\dot{P}^*, \tag{13}$$

and finally the equation for the *gas pressure*

$$P^* = P_m^* Z^{-3\gamma}. \tag{14}$$

## 3. Results

Eqs. (8) to (14) were solved for a number of nonlinear amplitudes from the range $1 < A \leqq 10$ ($A = R_M/R_e$, and $R_e$ is the equilibrium bubble radius). The initial pressure, $P_m^*$, was determined from a relation [4]

$$P_m^* = A^{-3\gamma}. \tag{15}$$

The initial conditions for eqs. (8) to (10), and (13) were $Z(0) = 1$, $\dot{Z}(0) = 0$. The values of physical constants used in the computations were $\gamma = 4/3$, $p_\infty = 10^5$ Pa, $\varrho_\infty = 10^3$ kg m$^{-3}$, $c_\infty = 1450$ m s$^{-1}$, $B = 3 \cdot 10^8$ Pa, $n = 7$.

For the comparison of the models we have selected the significant wall positions $Z_{m1}$, $Z_{M2}$, and the time of the bubble compression $T_{zc1}$ (Fig. 1).

Computed variations of the first minimum wall radius, $Z_{m1}$, with the amplitude, $A$, are shown in Fig. 2. It can be seen that for a given amplitude, $A$, Rayleigh's bubble has the most violent oscillations, and Herring's bubble the most moderate. Gilmore's bubble, thanks to an increase in $C$ for large pressures, $P$, oscillates more violently than Herring's bubble.

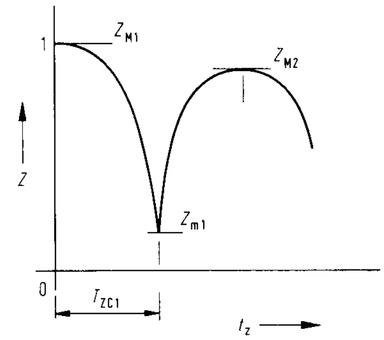


Fig. 1. Significant positions of the bubble wall in the compression system.

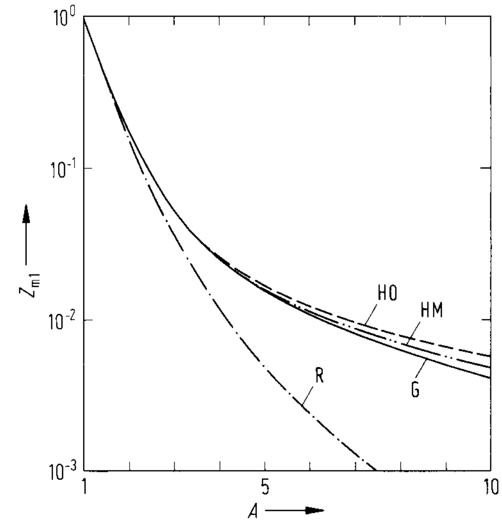


Fig. 2. Variation of the first minimum bubble wall radius, $Z_{m1}$, with the amplitude, $A$:
(——) Gilmore's model,
(– – –) Herring's original model,
(–·–) Rayleigh's model,
(–··–) Herring's modified model.

Similar behaviour can be traced in Fig. 3, where computed variations of the second maximum wall radius, $Z_{M2}$, with the amplitude, $A$, are displayed. Because Rayleigh's model of medium-sized bubbles does not account for any dissipative mechanism, it yields undamped oscillations and hence $Z_{M2} = Z_{M1} = 1$ for all $A$. On the other hand Herring's model, thanks to the lowest velocity of sound in the liquid, shows the largest radiation damping.

For a more detailed comparison of the models it is convenient to use a relative deviation, $\delta$, defined by a relation

$$\delta = \frac{M-N}{N} 100\,[\%]\,, \qquad (16)$$

where the $M$ stands for values to be compared and $N$ for reference values.

As an exact equation of the bubble-wall motion is not known at present, Gilmore's model, which is generally considered to give the most satisfactory results [5, 7], is used as a reference model here.

Computed variations of $\delta_{Zm}$ (i.e. of $Z_{m1}$) and $\delta_{ZM}$ (i.e. of $Z_{M2}$) with $A$ are shown in Figs. 4 and 5, respectively.

The differences found in the value of $T_{zc1}$ for all the models considered were negligible and hence we do not give the corresponding graphs here. Let us only note that the maximum computed deviation in Rayleigh's model was $(\delta_T)_{max} = -0.53\%$, in Herring's original model it was $(\delta_T)_{max} = -0.02\%$ and finally in the case of Herring's modified model we have found that $(\delta_T)_{max} = -0.49\%$.

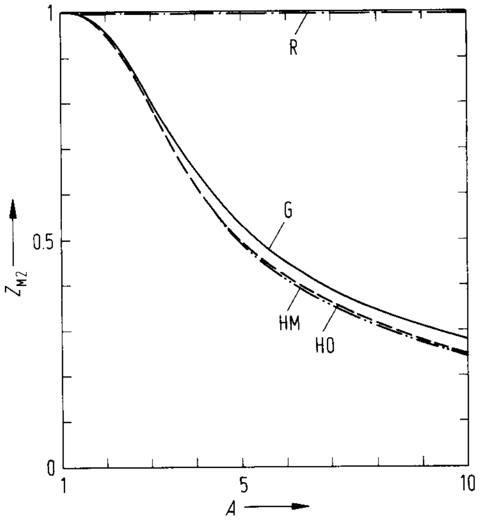


Fig. 3. Variation of the second maximum bubble wall radius, $Z_{M2}$, with the amplitude, $A$:
(——) Gilmore's model,
(- - -) Herring's original model,
(-·-) Rayleigh's model,
(-··-) Herring's modified model.

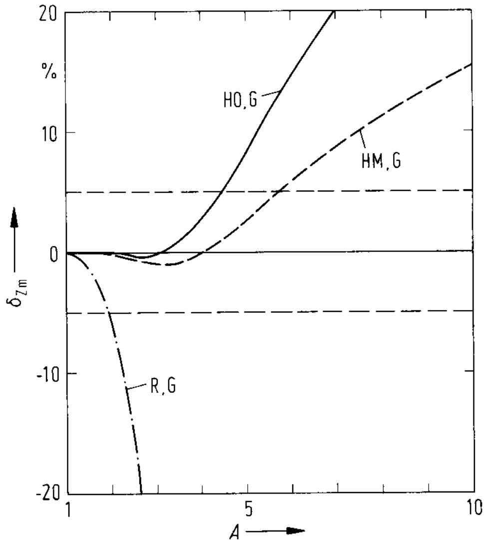


Fig. 4. Variation of the relative deviation, $\delta_{Zm}$, with the amplitude, $A$. Deviation between:
(——) Herring's original model and Gilmore's model,
(- - -) Herring's modified model and Gilmore's model,
(-·-) Rayleigh's and Gilmore's models.

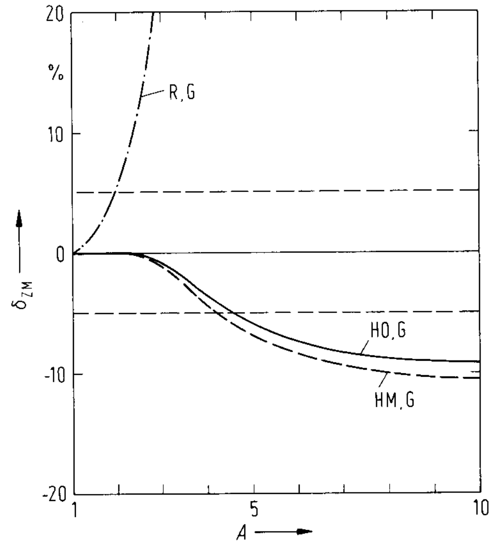


Fig. 5. Variation of the relative deviation, $\delta_{ZM}$, with the amplitude, $A$. Deviation between:
(——) Herring's original model and Gilmore's model,
(- - -) Herring's modified model and Gilmore's model,
(-·-) Rayleigh's and Gilmore's models.

## 4. Discussion

Let us define the area of the model validity as a range of amplitudes, for which the relative deviations satisfy unequality $\delta < 5\%$ for any quantity of interest. Then it follows from Figs. 4 and 5 that Rayleigh's model can be used for amplitudes up to $A \leqq 2$, and Herring's model up to $A \leqq 4.5$. An interesting finding is that the modified Herring model yields almost the same precision of computations as the original Herring model. This indicates a relative unimportance of various correction terms in the equation of motion and it is also a reason why we expect little difference between the models of Herring, Keller and Kolodner, Flynn, and Tomita and Shima.

In this paper Gilmore's model was used as a reference, though it is hard to assess at present how exact this model really is. Unfortunately, the only valid check, i.e. a comparison with experimental data, is impossible because there is none available. Thus the only criterion available is the result of computations performed by Hickling and Plesset [5]. These authors solved simultaneously Gilmore's model and the governing hydrodynamical equations for several values of $\gamma$, $p_\infty$, and $P_m$. The governing equations were integrated by a finite difference method and were assumed to provide an "exact" solution. From the graphs for $\gamma = 1.4$, $p_\infty = 10^5$ Pa, and $P_m = 10^4$, $10^3$, $10^2$, and 10 Pa published in [5] it is possible to infer that both methods yield essentially the same value of $Z_m$ for $P_m = 10^4$ and $10^3$ Pa ($A = 1.73$ and 3, respectively). For $P_m = 10^2$ Pa ($A = 5.18$) the relative deviation of Gilmore's result from the "exact" solution is approximately $\delta_{Zm} =$ 1.75%, and for $P_m = 10$ Pa ($A = 8.96$) it is $\delta_{Zm} =$ 6.7%. These figures indicate a degree of uncertainity inherent in the graphs shown in Figs. 4 and 5. However, the question still remaining to be answered is how correct are the "exact" results themselves.

The areas of validity presented above were determined from a comparison of the quantities $Z_{m1}$ and $Z_{M2}$. Evidently, for other quantities other limits on $A$ would be found. For example, it was noted in [4] that parameters of the radiated pressure pulses (the peak pressure in the wave and the effective pulsewidth) are very sensitive to changes in the bubble wall motion. Hence, limiting amplitudes that would be found from deviations of these quantities should be appropriately lower.

The limiting deviation $\delta = 5\%$ was selected quite arbitrarily (for other values of $\delta$ the areas of model validity can also be easily determined from Figs. 4 and 5). However, let us note here that there is no sense in attempting an extraordinary precise computation and that a 5% inaccuracy represents a reasonable compromise. The reason for this is that all the known bubble models assume ideal conditions that are never met in real situations. For example, the bubble shape is usually spherical only in the vicinity of the first maximum radius $R_{M1}$. On later occasions the shape may be seriously distorted, especially in the vicinity of the first minimum radius $R_{m1}$ (see, e.g. photographs in [2] and [10]).

Another question worth raising is the credibility of the bubble models under consideration. As we have said above, the validity of the models has not been verified experimentally yet (with the exception, perhaps, of linear oscillations). On the contrary, the few existing experimental works (see, e.g. [2, 11–13]) indicate serious discrepancies between the calculated and measured data. Hence, to stress the approximate nature of the models we deliberately use round values for the physical constants (e.g. $\varrho_\infty = 10^3\,\mathrm{kg\,m^{-3}}$ for water at 20 °C instead of the table value $\varrho_\infty = 998.23\,\mathrm{kg\,m^{-3}}$, etc.).

The solutions of the bubble oscillations were carried out for amplitudes as large as $A = 10$. From the mathematical point of view there is apparently no upper bound on the value of the parameter $A$ (e.g. Flynn [7] performed calculations for $A = 8.56$, Hickling and Plesset [5] for $A = 8.96$, Ebeling [14] for $A = 25$, and finally Lastman and Wentzell [9] for an amplitude as large as $A = 73.4$), the only limiting factor probably being the machine time available for numerical solution.

However, there seem to be certain physical constraints. As far as the behaviour of real bubbles is concerned, these physical limitations may originate, for example, in the nature of excitation techniques [15] or in the instability of a fast moving liquid-gas interface [16].

But it is also necessary to bear in mind that the models were derived under certain assumptions. For example, eq. (7) assumes a perfect gas, a uniform pressure field in the bubble interior, and a reversible change of the gas state. Evidently, these assumptions are only viable in the case of bubbles oscillating with moderate amplitudes.

It is our opinion (reasons for this opinion we want to discuss in greater detail elsewhere) that the maximum amplitude the bubble can oscillate with is much smaller than $A = 10$. If this assumption proves to be correct, then the results presented here indicate that as far as the bubble wall motion is concerned, Herring's model gives satisfactory results for almost all the amplitudes of interest. However, Gilmore's model remains irreplaceable in the studies of the shock waves and strong pressure pulse propagations.

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## References

[1] Rayleigh, J. W., On the pressure developed in a liquid during the collapse of a spherical cavity. Philos. Mag. **34** [1917], 94.

[2] Cole, R. H., Underwater explosions. Princeton University Press, Princeton 1948.

[3] Knapp, R. T., Daily, J. W., and Hammitt, F. G., Cavitation. McGraw-Hill, New York 1970.

[4] Vokurka, K., On Rayleigh's model of a freely oscillating bubble. I. Basic Relations. Czech. J. Phys. **B35** [1985], 28. II. Results. Czech. J. Phys. **B35** [1985], 110. III. Limits. Czech. J. Phys. **B35** [1985], 121.

[5] Hickling, R. and Plesset, M. S., Collapse and rebound of a spherical bubble in water. Phys. Fluids **7** [1964], 7.

[6] Keller, J. B. and Kolodner, I. I., Damping of underwater explosion bubble oscillations. J. Appl. Phys. **27** [1956], 1152.

[7] Flynn, H. G., Cavitation dynamics. I. A mathematical formulation. J. Acoust. Soc. Amer. **57** [1975], 1379. II. Free pulsations and models for cavitation bubbles. J. Acoust. Soc. Amer. **58** [1975], 1160.

[8] Tomita, Y. and Shima, A., On the behaviour of a spherical bubble and the impulse pressure in a viscous compressible liquid. Bull. JSME **20** [1977], 1453.

[9] Lastman, G. J. and Wentzell, R. A., Comparison of five models of spherical bubble response in an inviscid compressible liquid. J. Acoust. Soc. Amer. **69** [1981], 638.

[10] Lauterborn, W., Laser-induced cavitation. Acustica **31** [1974], 51 (In German).

[11] Arons, A. B. and Yennie, D. R., Energy partition in underwater explosion phenomena. Rev. Mod. Phys. **20** [1948], 519.

[12] Vokurka, K., Experimental study of the bubble pulse. In: Proceedings of the 7th Conference of the Czechoslovac Physicists, Prague 1981, paper 09-07 (In Czech).

[13] Hentschel, W. and Lauterborn, W., Acoustic emission of single laser-produced cavitation bubbles and their dynamics. Appl. Sci. Res. **38** [1982], 225.

[14] Ebeling, K. J., The behaviour of spherical, laser-produced cavitation bubbles in water. Acustica **40** [1978], 229 (In German).

[15] Vokurka, K., Excitation of gas bubbles for free oscillations. J. Sound Vib. (to appear).

[16] Plesset, M. S., Bubble dynamics and cavitation erosion. In: Finite-amplitude wave effects in fluids, L. Bjørno, ed., IPC Science and Technology Press, Guildford 1974, p. 203.