ON RAYLEIGH'S MODEL OF A FREELY OSCILLATING BUBBLE. II. RESULTS

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In the paper free bubble oscillations are examined in detail for a broad range of amplitudes. In the case of linear oscillations and of an empty bubble several interesting results are derived analytically. For a general value of the amplitude the equation of motion is solved numerically. Finally, the most important independent and scaling functions are presented.

1. INTRODUCTION

In the previous paper [1] the basic equations governing the oscillations of Rayleigh's bubble and describing the pressure and velocity fields in the surrounding liquid were derived. In this paper the bubble behaviour will be examined for a broad range of amplitudes. First, in section 2, linear oscillations will be analysed. An empty bubble will be studied in section 3. For a general value of the amplitude the equation of motion must be integrated numerically. Computed time histories of the bubble wall motion, pressure field in the liquid, and energies will be presented in section 4. Finally, in section 5, independent and scaling functions that represent a dependence of selected significant quantities on the amplitude A and on the adiabatic exponent γ will be given.

2. LINEAR OSCILLATIONS

For sufficiently small amplitudes the bubble oscillations can be considered to be approximately linear. An equation of motion of a linearly oscillating bubble is obtained best by passing in eq. (I-17) to the X variables and linearizing the resulting expression. One obtains ($\tau_x = t/[R_e(\varrho/p_\infty)^{1/2}]$, $X = (R - R_e)/R_e$)

$$\ddot{X} + 3\gamma X = 0 . \tag{II-1}$$

This is a well-known linear second-order differential equation with constant coefficients. Its solution for the initial conditions $X(0) = A_x$ and $\dot{X}(0) = 0$ is a harmonic function ($A_x = X_M = (R_M - R_e)/R_e = A - 1$)

$$X = A_x \cos(\omega_{x0}\tau_x) , \tag{II-2}$$

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where ω_{xo} is the nondimensional circular frequency ($\omega_{xo} = \omega_o R_e(\varrho/p_\infty)^{1/2}$) given by the equation

(II-3) $$\omega_{xo}^2 = 3\gamma\,.$$

This relation was first derived by Minnaert [2]. The compression time, T_{xc}, can be determined from (II-3) as one half of the bubble oscillation period, T_{xo}, i.e.

(II-4) $$T_{xc} = \frac{\pi}{\sqrt{(3\gamma)}}\,.$$

Let us now consider a pressure wave radiated by the linearly oscillating bubble. For this purpose eq. (I-25) will also be transformed via the Y system into the X system. After linearization we obtain ($p_x = (p/p_\infty - 1)(r/R_e)$)

(II-5) $$p_x = \ddot{X} = -3\gamma X = -3\gamma A_x \cos(\omega_{xo}\tau_x)\,.$$

It follows from (II-5) that the larger the γ, the more intensively the bubble radiates sound. The bubble also radiates in opposition to the bubble wall motion, i.e. for $X > 0$ the pressure is negative ($p_x < 0$) and vice versa. From (II-5) a peak pressure in the wave equals

(II-6) $$p_{xp} = 3\gamma A_x\,.$$

This relation was first found by Strasberg [3].

The effective pulse width can be found by substituting (II-4), (II-5), and (II-6) into the definition formula (I-29). We obtain

(II-7) $$\vartheta_x = \frac{\pi}{\sqrt{(3\gamma)}}\,.$$

It can be seen that in the case of the linear oscillations the effective pulse width equals the compression time T_{xc}.

It can also be easily shown (e.g. by linearizing eq. (I-1)) that the acoustic pressure at the bubble wall is

(II-8) $$P_a^* = -3\gamma X = -3\gamma A_x \cos(\omega_{xo}\tau_x)\,.$$

Then, comparing (II-5) with (II-8) it can be verified that for a linearly oscillating bubble the pressure wave contains only the communicated component. This can also be seen directly by linearizing eq. (I-21).

The position independent velocity field in Rayleigh's model is given by eq. (I-27). Passing to the X system ($v_x = v(\varrho/p_\infty)^{1/2}(r/R_e)^2$) and linearizing the resulting equation we obtain

(II-9) $$v_x = \dot{X} = -A_x\omega_{xo}\sin(\omega_{xo}\tau_x)\,.$$

Hence, by comparing (II-5) with (II-9) it can be verified that in Rayleigh's model the velocity field is reactive.

Some authors (see, e.g., Minnaert [2]) compare an oscillating bubble with a simple oscillatory system, in which the liquid plays the role of the inert mass and the gas the role of the spring. This approach, though fully correct as far as the equation of motion is concerned, may give a misleading impression in regard to the energies involved. As we shall show in the following, the main flow of the energy in a bubble performing linear oscillations is not between the liquid and the gas, but between the potential and the internal energy (note that the internal energy of the gas is called the potential energy by Minnaert [2], Devin [10], and some other authors).

In the linear case the energies are given by

(II-10) $$\Delta E_{xp} = 3(A_x - X)\,,$$

(II-11) $$E_{xk} = \tfrac{3}{2}\dot{X}^2\,,$$

(II-12) $$E_{xi} = 3(A_x - X) - \tfrac{3}{2}\,3\gamma(A_x^2 - X^2)\,,$$

and the energy relation is

(II-13) $$\Delta E_{xp} = E_{xk} + E_{xi}\,.$$

Substitution of (II-10)–(II-12) into (II-13) gives

(II-14) $$\dot{X}^2 = 3\gamma(A_x^2 - X^2)\,.$$

This differential equation can be easily solved and the solution naturally has the well-known form (II-2). When differentiating (II-14) with respect to time eq. (II-1) can also be obtained.

Let us examine the different terms in eq. (II-13). The kinetic energy has a maximum, $E_{xk\,max}$, for $X = 0$, $\tau_x = T_{xc}/2$. For $\gamma = 4/3$ it further follows that $E_{xk\,max} = 3A_x^2$. The internal energy of the gas has a maximum, $E_{xi\,max}$, when $\tau_x = T_{xc}$. Then $E_{xi\,max} = 6A_x$. For example, if $A_x = 0{\cdot}01$, $E_{xi\,max}$ is 200 times larger than $E_{xk\,max}$. With growing amplitude the share of $E_{xk\,max}$ also grows, but even for amplitude as large as $A_x = 0{\cdot}1$ the maximum kinetic energy is still 20 times smaller than the maximum internal energy.

It can be seen from (II-10)–(II-13) and from the foregoing discussion that during the first half of the compression phase the main flow of the energy is between E_{xp} and E_{xi} and only a minute part of E_{xp} is transformed into E_{xk}. The main flow of the energy into the gas goes on also in the second half of the compression phase; however, now it is accompanied by a slight stream of the energy supplied by the flowing liquid. Nevertheless, it is just this slight stream of energy that determines the basic features of the wall motion.

All relations given in this section hold for $A_x \to 0$. However, if the amplitude is being increased, the nonlinear character of the oscillations will soon become evident (approximately for $A_x > 0{\cdot}05$), primarily because of the nonsymmetry in the bubble wall motion [4].

3. EMPTY BUBBLE

While in the preceding section the case when $A \to 1$ was studied in detail, in this section we shall consider the other limit, namely the so-called empty bubble. In this case the pressure $P_m^* = 0$, which in the compression system formally corresponds to an infinite amplitude $A \to \infty$ (cf. eq. (I-15)). For such an amplitude the equation of motion (I-12) assumes the form

$$\ddot{Z}Z + \tfrac{3}{2}\dot{Z}^2 = -1\,. \tag{II-15}$$

From (I-11), the wall velocity is

$$\dot{Z}^2 = -\tfrac{2}{3}(1 - Z^{-3})\,. \tag{II-16}$$

Finally, from (I-13) the wall acceleration is given by

$$\ddot{Z} + Z^{-4} = 0\,. \tag{II-17}$$

Eq. (II-16) can be integrated in a closed form for Z ranging from 1 to 0. The result is

$$T_{zc} = \int_0^{T_{zc}} \mathrm{d}\tau_z = \int_0^1 \frac{\mathrm{d}Z}{[\frac{2}{3}(Z^{-3} - 1)]^{1/2}} = 0{\cdot}915\,. \tag{II-18}$$

This is the collapse time of the empty bubble, which was first found by Rayleigh [5].

After substituting (II-16) and (II-17) into (I-25) the pressure in the acoustic field of the empty bubble is given by

$$p_z = \tfrac{1}{3}Z(Z^{-3} - 4)\,. \tag{II-19}$$

From this expression it follows that $p_z = 0$ if $Z = Z_p = 0{\cdot}63$. Since the acoustic pressure at the bubble wall is constant and equals $P_a^* = -1$, the radiated pressure wave in the acoustic field contains only a dynamic component (the second term in (I-21)). However, this is not true for the moment of a complete collapse ($Z \to 0$). In this case, though the dynamic component disappears, the pressure p_z is infinite anywhere in the liquid.

The velocity field can be obtained from (I-28) and (II-16) in the form

$$v_z^2 = \tfrac{2}{3}Z(1 - Z^3)\,. \tag{II-20}$$

At the beginning of the collapse, when $Z = Z_M = 1$, the velocity field is zero. The particle velocity, v_z, has a maximum for $Z = Z_c = 0{\cdot}63$, i.e. at the same moment when the pressure field p_z equals zero. In the subsequent times the particle velocity decreases until, as $Z \to 0$, it equals zero for any $z > 0$ again.

As there is no restoring force, the empty bubble has only one phase, namely the so-called collapse phase (this designation is also used by some authors even for the compression phase of the gas bubbles). It follows from (II-16), (II-17) and (II-19) that as $Z \to 0$, the wall velocity, $\dot{Z}$, acceleration, $\ddot{Z}$, and the pressure, p_z, are infinite, i.e. $\dot{Z} \to -\infty$, $\ddot{Z} \to \infty$, and $p_z \to \infty$. Though the liquid velocity is zero outside the collapse point at this time, the kinetic energy of the liquid remains finite ($E_{zk} \to 1$) and is concentrated just in the bubble centre. This singularity was of great concern in the past (see, e.g., Hunter [6]).

Even if real bubbles cannot be completely empty, the model considered here is, owing to its simplicity, very important. First, the behaviour of a gas bubble having a sufficiently large amplitude A approaches asymptotically in many respects the behaviour of the empty bubble. This is true primarily for the initial stages of the compression when the gas pressure is small. As will be shown later, the difference is very small already for bubbles oscillating with amplitudes $A \geqq 2$. The compression time, T_{zc}, of the gas bubble with growing amplitude also very quickly approaches the asymptotic value 0·915, so that this value is often used to approximate the actual compression time.

Second, there is an important class of the so-called vapour bubbles. These model bubbles are supposed to contain only the vapour of the surrounding liquid. It is also assumed that the processes of condensation and evaporation in such a bubble can go off with an infinite velocity, so that the pressure inside the vapour bubble is constant at all times. It equals P_v, the liquid vapour pressure at a given temperature. In this case the corresponding equations are the same as those given above (i.e. eqs. (II-15)–(II-20)), the only difference being in the definition of nondimensional time, where the ambient pressure, p_∞, is now replaced by the term $(p_\infty - P_v)$.

As in no real bubble can the processes of evaporation and condensation go off with an infinite velocity, the vapour bubble represents only a useful abstraction. It can be expected that in real bubbles the vapour begins to behave as a superheated gas in the final stages of the collapse, when the wall velocity becomes too high. The vapour bubble can then be modelled to a first approximation by a gas bubble having sufficiently large amplitude. Thus, an analysis given in the following sections can also be applied to the vapour bubbles.

4. NONLINEAR OSCILLATIONS

In sections 2 and 3 two important limiting cases, i.e. the linear oscillations, which represent the small amplitude limit ($A \to 1$), and the empty bubble, which formally corresponds to the infinite amplitude ($A \to \infty$), have been studied. These two limiting cases had one very important common feature, namely the relative simplicity that allowed us to find several interesting results analytically. In this section the bubble oscillations for a general value of the amplitude will be considered. For such oscillations the solution of the equation of motion can be obtained only numerically, e.g. by the Runge-Kutta method. Several examples of computed wall motions are given in fig. II-1.

It can be seen in fig. II-1 that for small amplitudes the wall motion is a harmonic function of time. However, owing to the inherent nonlinearity in the bubble wall motion it becomes appreciably disturbed for slightly larger amplitudes.

In fig. II-1 the curve $Z(\tau_z)$ for $A \to \infty$ represents a certain boundary, because no bubble can be compressed faster than Rayleigh's empty bubble. All other solutions $Z(\tau_z)$ must therefore lie to the right of this curve. The other boundary is formed by the time axis ($Z = 1$), whose section $(0; \pi/\sqrt{(3\gamma)})$ represents a solution for $A \to 1$. All other solutions of the equation of motion must lie, regardless of the model used, amplitude A, and exponent γ, between these two boundaries.

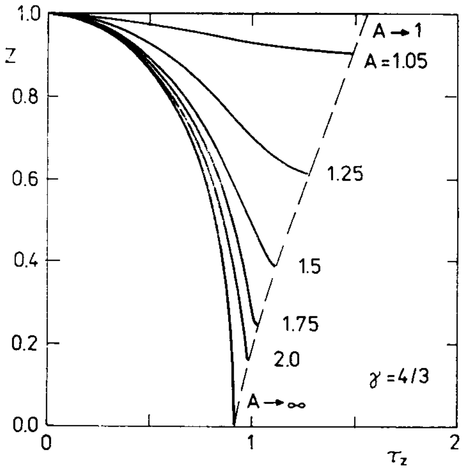

Fig. II-1. Time histories of the bubble wall motions for different amplitudes.

It can be shown by comparing compressible and noncompressible models that a solution $Z(\tau_z)$ for $A = 2$ in Rayleigh's model is still within a reasonable accuracy. Therefore all the curves $Z(\tau_z)$ for amplitudes $2 < A < \infty$ must lie between the curves $Z(\tau_z)$ corresponding to $A = 2$ and $A \to \infty$, regardless of the model used. Hence, it can be concluded that the form of the curve $Z(\tau_z)$ for $A > 2$ will vary only little with the change of the amplitude, except for the value of Z_m, where Z_m shifts downwards with increasing A (see fig. II-1). As a recording of Z_m by high-speed cinematographic methods is extremely difficult [7] it follows then from the foregoing discussion that there is a certain limit regarding the amount of information to be gained by photographic studies of the bubble dynamics.

From the solution $Z(\tau_z)$ a number of other functions can be determined. For example, eq. (I-1) can be used to find a variation of the pressure in the bubble interior with time. The temperature of the gas can be determined from the relation $\Theta = \Theta_m Z^{-3(\gamma-1)}$, where Θ_m is the temperature in the gas when $Z = Z_M = 1$. This temperature can be computed from the initial temperature Θ_M using the relation $\Theta_m = \Theta_M W_M^{-3(\gamma-1)} = \Theta_M Z_m^{3(\gamma-1)}$.

Similarly, knowledge of $Z(\tau_z)$ enables determining the velocity field from eq. (I-27) or (I-28), and the pressure field from eq. (I-24) or (I-25). As the time dependence of the pressure at a given point in the acoustic field is often measured experimentally with a hydrophone, it is interesting to compute for the purpose of comparison the theoretical waveforms $p_z(\tau_z)$. Several examples of the computed waves are given in fig. II-2. It can be seen again that for a small amplitude the pressure approaches a harmonic function of time. However, if the amplitude is being increased, the peak value grows rapidly and the width of the positive pressure decreases so that the form of the wave starts to resemble a pulse very soon. In fact, the peak pressure, p_{zp}, and the effective pulse width, ϑ_z, are very sensitive to the change of the amplitude and thus they seem to be suitable for determining the unknown bubble parameters experimentally.

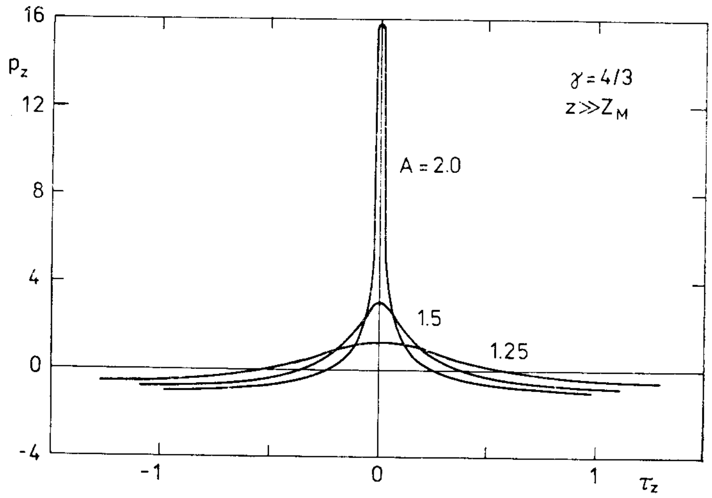

Fig. II-2. Forms of pressure pulses for different amplitudes.

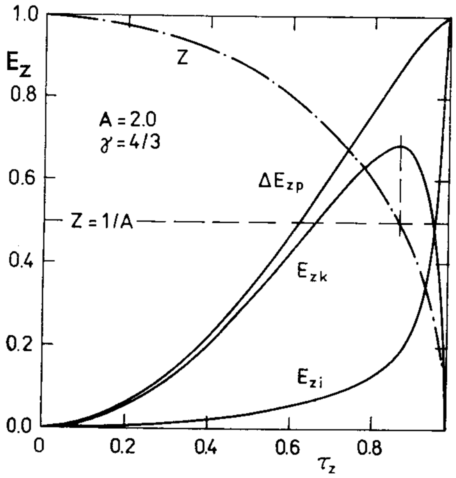

Fig. II-3. Time dependence of energies during the bubble compression.

Finally, the function $Z(\tau_z)$ can be used to find the time dependence of energies from eqs. (I-7)–(I-9). An example of the computed curves $E_z(\tau_z)$ is given in fig. II-3. It can be seen that during the compression phase the potential energy of the bubble transforms first to the kinetic energy of the liquid and then in the final stages of the compression (after the wall has passed the equilibrium position) the kinetic energy of the liquid is transformed into the internal energy of the gas. This transfer of the potential energy into the internal via the kinetic energy is even more pronounced for larger amplitudes. For example, in the linear case the kinetic energy was almost negligible. If $A = 2$, then $E_{zk\,max}$ represents approximately 70% of the full energy and for $A \to \infty$ all the potential energy is transformed into kinetic energy. It also follows for bubbles oscillating with larger amplitudes that in the interval (Z_M, Z_e) the motion of the wall is determined primarily by the inertial forces and hence there is little difference between vapour, empty, and sufficiently violently oscillating gas bubbles. This fact has been already mentioned in the preceding section.

5. INDEPENDENT AND SCALING FUNCTIONS

It is usually not necessary to have such complete information about the bubble wall motion or radiated pressure wave as is given by a solution $Z(\tau_z)$ or $p_z(\tau_z)$, respectively. In many cases it is sufficient to know just some significant values of these solutions and of associated quantities. In this section we shall consider in greater detail the independent quantities P^*_M, P^*_m, and $\dot{Z}_{max}$, and the scaling quantities Z_m, T_{zc}, p_{zp}, and ϑ_z. In particular, we shall determine their dependences on A and γ. These dependences will be called the independent and the scaling functions.

The designation "independent" means independence of the bubble size here (in the frame of the present theory) and should not be confused with the designation "independent variable" used in the usual mathematical sense. The designation "scaling" is used to stress that the respective functions and quantities will be the same for bubbles of different sizes, if the scales by which they are measured are changed by the same factor as the characteristic bubble dimensions.

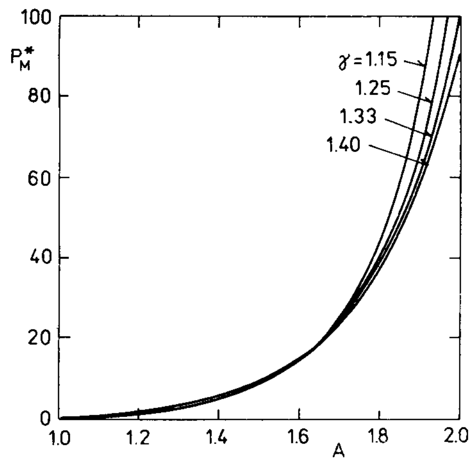

Fig. II-4. Independent functions $P^*_M(A, \gamma)$.

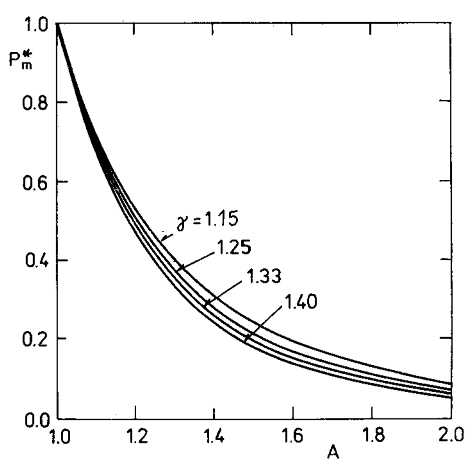

Fig. II-5. Independent functions $P^*_m(A, \gamma)$.

The first independent functions to be considered here are $P_M^* = P_M^*(A, \gamma)$ and $P_m^* = P_m^*(A, \gamma)$. Using eq. (I-1) and the definition of the amplitude A we have

(II-21) $$P_M^* = A^{-3\gamma} Z_m^{-3\gamma},$$

and

(II-22) $$P_m^* = A^{-3\gamma}.$$

The functions P_M^* and P_m^* are displayed in figs. II-4 and II-5. These graphs can be conveniently used to relate the different intensity measures, i.e. P_M^*, P_m^*, and A. The value of γ was chosen to cover both adiabatic (diatomic and triatomic gases, and the products of explosions [8]) and polytropic models.

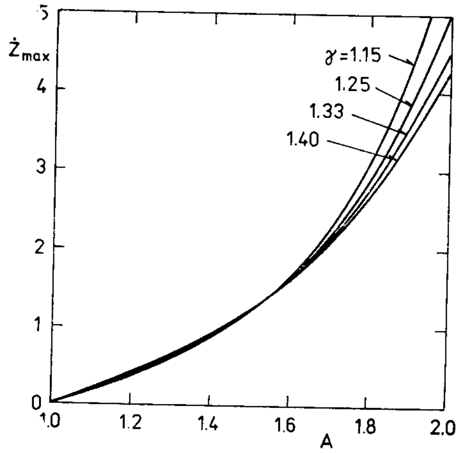

Fig. II-6. Independent functions $\dot{Z}_{max}(A, \gamma)$.

Dependence of the maximum wall velocity $\dot{Z}_{max}$ on A and γ represents another important independent function. This function is displayed in fig. II-6. It can be seen that for $A \leqq 2$ the maximum wall velocities, $\dot{Z}_{max}$, are rather small in comparison with the velocities of sound in air and water. Thus the assumptions of the uniform pressure field in the bubble interior, and of the noncompressible liquid, are not violated too crudely for the given range of amplitudes.

Calculated scaling functions $Z_m = Z_m(A, \gamma)$ and $T_{zc} = T_{zc}(A, \gamma)$ are given in figs. II-7 and II-8. The variable Z_m represents a basis for computation of several other significant quantities, e.g. P_M^* and p_{zp}, and its determination was considered in great detail in the previous work [1]. The quantity T_{zc} is often used to determine the bubble size in experiments. It can be found analytically for $A \to 1$ (eq. (II-4)) and for $A \to \infty$ (eq. (II-18)). For other values of A it must be computed numerically. From fig. II-8 it can be seen that T_{zc} approaches the asymptotic value 0·915 rather quickly. It is possible to show that for $A \geqq 3$ the deviation of T_{zc} from the asymptotic value is insignificant even for models considering the liquid compressibility. It is this "independence" of T_{zc} of A and γ that makes it possible to determine R_M just from the knowledge of T_c.

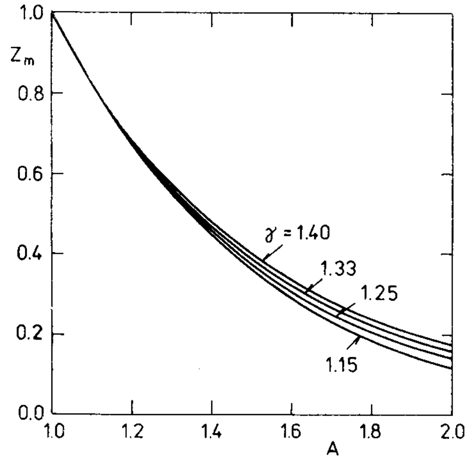

Fig. II-7. Scaling functions $Z_m(A, \gamma)$.

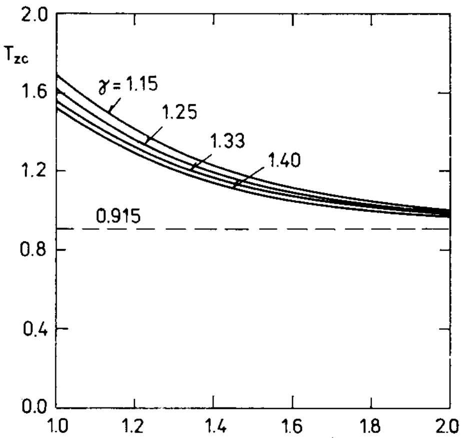

Fig. II-8. Scaling functions $T_{zc}(A, \gamma)$.

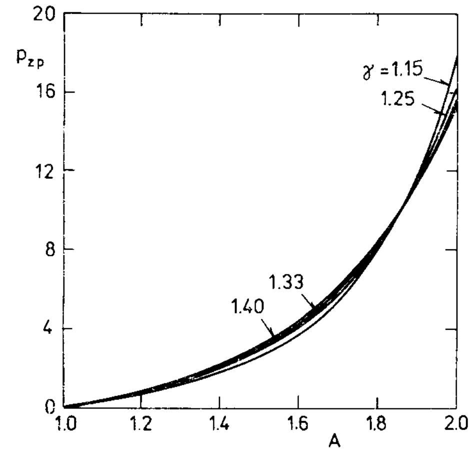

Fig. II-9. Scaling functions $p_{zp}(A, \gamma)$.

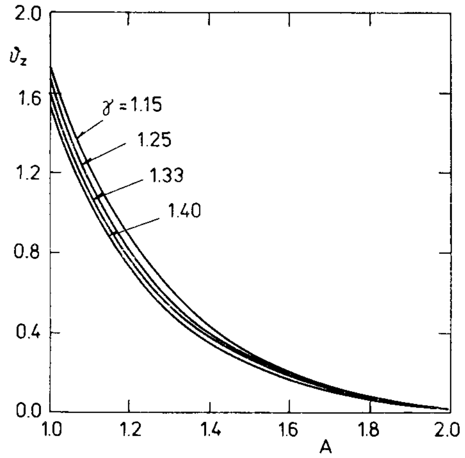

Fig. II-10. Scaling functions $\vartheta_z(A, \gamma)$.

The last two functions to be considered here are $p_{zp} = p_{zp}(A, \gamma)$ and $\vartheta_z = \vartheta_z(A, \gamma)$. To determine p_{zp}, eqs. (I-12) and (I-25) can be rearranged. As the peak pressure in the wave is radiated when $Z = Z_m$ and $\dot{Z} = 0$, we obtain

$$p_{zp} = Z_m(A^{-3\gamma}Z_m^{-3\gamma} - 1)\,. \tag{II-23}$$

The effective pulse width, ϑ_z, can be obtained by numerical integration from eq. (I-29). The calculated scaling functions p_{zp} and ϑ_z are given in figs. II-9 and II-10.

From figs. II-4, II-6, and II-9 it can be seen that for small amplitudes (approximately for $A < 1{\cdot}7$) bubbles with larger γ oscillate more intensively than bubbles with smaller γ. For amplitudes $A > 1{\cdot}7$ it is just the opposite: bubbles with smaller γ oscillate more violently. Finally, for $A \doteq 1{\cdot}7$ the bubble behaviour practically does not depend on γ.

6. DISCUSSION

In evaluating experimental data several functions defined and computed in this paper can be used. For example, when working with a hydrophone, quantities T_c, p_p (at a given distance r), and ϑ can be determined from the signal received. If functional dependences of these quantities on bubble parameters R_M, A, and γ are known, then these basic bubble parameters can be determined. No doubt, the work involved will be much simplified if the bubble under investigation is of medium size, because in such a case the scaling functions T_{zc}, p_{zp}, and ϑ_z that were determined in section 5 can be used directly.

Since Rayleigh's model considers the liquid to be noncompressible, it gives satisfactory results only for sufficiently small amplitudes. The range of usable amplitudes can be determined, for example, by comparing results of Rayleigh's and Gilmore's [9] models. In this way we have found that the time of compression T_c is only very little influenced by the compressibility (the difference in T_{zc} in these two models is only 0·5% for $A = 2$). However, the difference in the minimum radius Z_m is 1·1% for $A = 1{\cdot}5$ and 5·6% for $A = 2$. As can be expected, the peak pressure, p_{zp}, is the most sensitive quantity. In this case the difference is as much as 4·1% for $A = 1{\cdot}5$ and 19·4% for $A = 2$. All the computations mentioned were done for $\gamma = 4/3$.

Though no exact data are available at the present time it can be expected that explosion generated bubbles are excited during the growth phase to larger amplitudes than considered here [1, 11]. Hence the quantitative results we have obtained in this paper can be used only for later oscillations when the bubble wall motion is sufficiently damped down. However, even in evaluating the earlier phases of the bubble life Rayleigh's model can yield invaluable services because thanks to its simplicity it enables a qualitative analysis not possible with more exact models. Thus it helps to obtain the necessary insight into the processes involved, and the results presented here can be used as a basis for further work with more complicated models considering the liquid compressibility.

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