ON RAYLEIGH'S MODEL OF A FREELY OSCILLATING BUBBLE. I. BASIC RELATIONS*)

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Free nonlinear oscillations of a gas bubble in a noncompressible liquid are analysed. Excitation o the growth is considered in the form of an instantaneous delivery of heat into the bubble nterior. The characteristic features of the pressure and velocity fields in the liquid are discussed.

1. INTRODUCTION

The first bubble model was described in 1917 by Rayleigh [1], who used an assumption of a noncompressible liquid. Though more exact bubble models were published later [2–6], Rayleigh's model has preserved its significance. This is, first of all, because of its simplicity that often makes it possible to obtain a better insight into the processes involved in bubble oscillations.

Rayleigh [1] considered primarily an empty bubble. He also indicated a solution in the case of a gas bubble; however, he did not pursue this direction in greater depth. Rayleigh's model was further elaborated by Lamb [7], Minnaert [8], Taylor [2], Plesset [9, 10], Noltingk and Neppiras [11], Poritsky [12], Robinson and Buchanan [13], and Güth [14, 15]. Recently, Rayleigh's model was studied by Shima nd Tomita [16], Shima and Tsujino [17], Lauterborn [18], Prosperetti [19, 20], and Samek [21, 22].

In spite of the great number of works on Rayleigh's model its potentialities have not been by far exhausted. In this paper we give some new results regarding free oscillations of gas bubbles. The work was motivated by an effort to explain experimental results obtained in a study of the pressure pulses emitted by laser generated bubbles [23]. Therefore, the emphasis is on nonlinear oscillations of medium-sized bubbles and on waves radiated into the liquid during their lives, though other cases will also be included.

2. PRELIMINARY REMARKS

The term "Rayleigh's model" (also "Rayleigh's bubble") is used here to denote a bubble in a noncompressible liquid. We shall study free oscillations of a spherical Rayleigh's bubble situated in an infinite volume of a liquid and oscillating in a zero

*) This work and the following papers [24, 28] are taken from the author's Ph. D. dissertation [33]. In shortened form these papers were also presented at the seminar on cavitation [34].

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mode, i.e. radially pulsating. The bubble is supposed to be large enough for the surface tension, viscosity and heat conduction to be neglected. On the other hand it is supposed to be sufficiently small for gravity to be omitted from the analysis. These effects will be taken into account briefly later [24].

Let the bubble be filled with a perfect gas whose amount does not vary during the bubble life (the gas bubble). The bubble is supposed to be initially at rest and to have a radius R_m. The pressure and temperature in the liquid at infinity are constant and equal to p_∞ and Θ_∞, respectively. At the time $t = 0$ heat ΔQ is supplied to the bubble interior. As a consequence, the pressure and temperature of the gas will rise instantaneously to values $P_M > p_\infty$, $\Theta_M > \Theta_\infty$. The excess pressure, $P_M - p_\infty$, will cause the bubble to grow to a maximum radius, R_M, where the pressure will have a minimum value P_m. It will be supposed that no heat flows across the interface (wall) and that both the pressure and the temperature fields in the bubble interior are uniform. At R_M the direction of the bubble wall motion reverts and the bubble will oscillate between R_M and R_m (fig. I-1). As liquid compressibility, gravity, viscosity and heat conduction are not considered here, the oscillations are undamped.

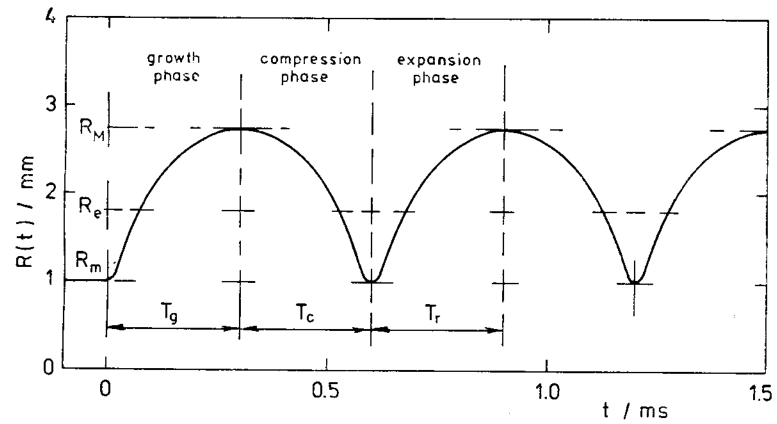

Fig. I-1. Time history of the bubble wall position. $P_M = 1$ MPa, $\gamma = 4/3$, $R_m = 1$ mm.

During its life a bubble goes through several phases. In the case considered here the wall motion starts with an expansion and this expansion phase lasts a time T_g. It is followed by a compression (also contraction or collapse) phase of duration T_c. Whereas the first expansion is also called the growth phase, the later expansions are sometimes called the rebound phases.

The dynamic motion of the bubble wall is accompanied with a change of the pressure in the surrounding liquid. The form of a radiated wave is depicted in fig. I-2. When the wall is near the maximum radius, R_M, the bubble radiates a wave of rarefaction and in the vicinity of R_m a wave of compression. The wave of compression is usually referred to as the bubble pulse [2].

In this work variables corresponding to the bubble wall will be denoted by uppercase letters and those corresponding to other points by lower-case letters. Differentiation with respect to time will be denoted by dots.

To reduce the number of parameters and to obtain results in a more general form nondimensional quantities and a normalized form of equations will be systematically used [25].

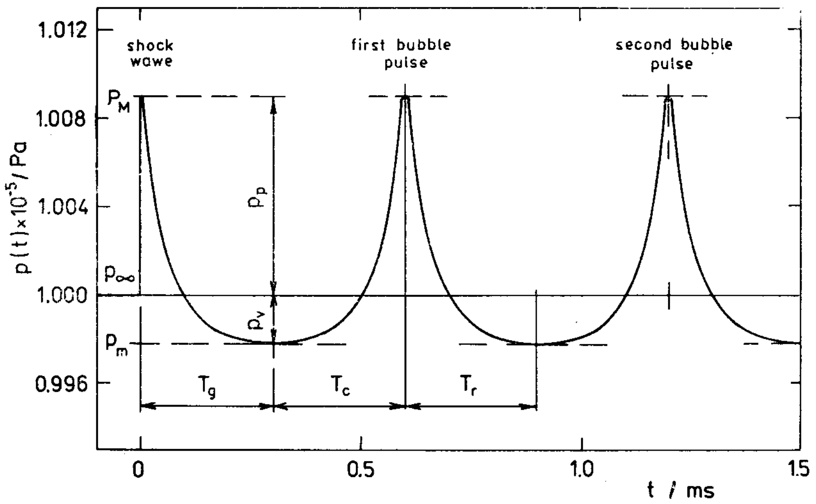

Fig. I-2. Pressure wave radiated by Rayleigh's bubble. $P_M = 1$ MPa, $\gamma = 4/3$, $R_m = 1$ mm, $r = 0{\cdot}1$ m.

There are basically two kinds of the nondimensional variables and parameters entering into the normalized equations. First of all, there are nondimensional quantities, whose pi groups do not contain a characteristic bubble dimension. These nondimensional variables will be denoted by an asterisk.

In the second case the pi groups contain a characteristic bubble dimension, as, for example, radii R_M, R_e or R_m. It is advantageous to use for different tasks these quantities defined with different characteristic bubble dimensions. Here we shall occasionally use four such systems of the nondimensional quantities and normalized equations: (1) an expansion W system, which is based on the radius R_m, (2) a compression Z system, which uses the radius R_M, (3) an equilibrium Y system, which uses an equilibrium radius R_e, and (4) a linear X system, which is also based on the radius R_e. The difference between these four systems is not only formal, as might seem at first sight, but has physical foundations in different excitation techniques [26]. The nondimensional quantities belonging to this second group will be denoted either by letters or subscripts w, z, y and x.

In this work all computations are done for water under ordinary laboratory conditions. The following values of parameters and physical constants are used:

Pressure at infinity	p_∞	$= 10^5$ Pa ,
Density of water	ϱ	$= 10^3$ kg m^{-3} ,
Acceleration of gravity	g	$= 10$ m s^{-2} ,
Surface tension	σ	$= 72 \times 10^{-3}$ N m^{-1} ,
Viscosity	η	$= 10^{-3}$ kg s^{-1} m^{-1} .

3. EQUATION OF MOTION

Though the equation of motion of Rayleigh's model is well known, it will be derived here step by step to enable references later in the text.

Let us consider a bubble that was excited in a certain way to oscillate [26]. As most interesting phenomena associated with the bubble oscillations occur during final stages of the compression and early stages of the expansion, our analysis will start at the instant the bubble reaches its maximum radius R_M and begins to contract.

Let us therefore suppose that at $t = 0$ the radius is $R = R_M$, the wall and non-compressible liquid are at rest, and the pressure inside the bubble is $P_m < p_\infty$. Owing to the excess pressure, $p_\infty - P_m$, the bubble wall begins moving inward and the gas is being compressed. As no heat flows across the interface, the compression is adiabatic and hence it holds that

$$P = P_m \left(\frac{R_M}{R}\right)^{3\gamma} . \tag{I-1}$$

Here γ is the ratio of the specific heats, $\gamma = c_p/c_v$. The work done in compressing the gas during the wall motion from R_M to R equals ($E_i = -\int P \, dV$)

$$E_i = \tfrac{4}{3}\pi \frac{1}{\gamma - 1} P_m R_M^3 \left[\left(\frac{R_M}{R}\right)^{3(\gamma-1)} - 1\right] . \tag{I-2}$$

Since the fluid is incompressible, the velocity v at a point r is

$$v = \dot{R} \frac{R^2}{r^2} , \tag{I-3}$$

and the kinetic energy of the liquid equals ($E_k = \frac{1}{2} \int v^2 \, dm$)

$$E_k = 2\pi\varrho \dot{R}^2 R^3 , \tag{I-4}$$

where ϱ is the liquid density. The initial potential energy of the bubble is $E_{pM} = (\frac{4}{3})\pi p_\infty R_M^3$. During the compression from R_M to R the decrease in the potential energy equals

$$\Delta E_p = \tfrac{4}{3}\pi p_\infty (R_M^3 - R^3) . \tag{I-5}$$

Finally, the bubble compression is governed by an energy relation

$$\Delta E_p = E_k + E_i . \tag{I-6}$$

At this point it is advantageous to introduce the nondimensional compression variables. They are: time $\tau_z = t/[R_M(\varrho/p_\infty)^{1/2}]$, bubble radius $Z = R/R_M$, energy $E_z = E/E_{pM}$ and pressure $P^* = P/p_\infty$. Then eqs. (I-2), (I-4)–(I-6) in the compression system have the forms

$$E_{zi} = \frac{1}{\gamma - 1} P_m^*[Z^{-3(\gamma-1)} - 1], \tag{I-7}$$

$$E_{zk} = \tfrac{3}{2}\dot{Z}^2 Z^3, \tag{I-8}$$

$$\Delta E_{zp} = 1 - Z^3, \tag{I-9}$$

$$\Delta E_{zp} = E_{zk} + E_{zi}. \tag{I-10}$$

Substituting (I-7)–(I-9) into (I-10) we obtain

$$\dot{Z}^2 = \tfrac{2}{3}Z^{-3}\left[1 - Z^3 - \frac{1}{\gamma - 1} P_m^*(Z^{-3(\gamma-1)} - 1)\right]. \tag{I-11}$$

The initial condition of this differential equation is $Z(0) = 1$. Analytical solution of (I-11) has not been found yet. Unfortunately, this equation cannot be easily integrated numerically either, because for $Z(0) = 1$ it holds that $\dot{Z}(0) = 0$ and the usual methods (e.g. the Runge-Kutta method) cannot be applied. Differentiation of (I-11) with respect to time gives

$$\ddot{Z}Z + \tfrac{3}{2}\dot{Z}^2 = P_m^* Z^{-3\gamma} - 1. \tag{I-12}$$

Finally, substituting (I-11) into (I-12) we also have

$$\ddot{Z} = Z^{-4}\left[-1 + \frac{1}{\gamma - 1} P_m^*(\gamma Z^{-3(\gamma-1)} - 1)\right]. \tag{I-13}$$

Relations (I-12) and (I-13) are the equations of motion for the bubble wall in the compression system. The initial conditions are $Z(0) = 1$ and $\dot{Z}(0) = 0$. The pressure P_m^* represents a parameter that determines the intensity of the bubble oscillations in the compression system. However, to be able to compare results obtained in different systems, a nonlinear amplitude, A, defined as [26]

$$A = \frac{R_M}{R_e}, \tag{I-14}$$

will be preferred. Here R_e is an equilibrium radius defined by the condition that $P = p_\infty$ when $R = R_e$. Then from (I-1) it also follows that

$$P_m^* = A^{-3\gamma}. \tag{I-15}$$

Eq. (I-12) is the most often used relation in representing Rayleigh's model. Though in the modern literature on bubblelogy it is usually derived from hydrodynamic equations (see, e.g., [4, 9, 10, 16]), the derivation via the energies was preferred here. Each method of derivation certainly has its pros and cons. The advantage of the way used here is that one always obtains, as an intermediate result, the equation for the wall velocity. The other advantages will be obvious later when determining the limits of the analysis presented here [24].

The equation of motion in the expansion system can be determined in the same manner as above. Defining the nondimensional expansion variables as $\tau_w = t/[R_m(\varrho/p_\infty)^{1/2}]$ and $W = R/R_m$ we obtain [26]

$$\ddot{W}W + \tfrac{3}{2}\dot{W}^2 = P^*_M W^{-3\gamma} - 1\,, \tag{I-16}$$

with initial conditions $W(0) = 1$ and $\dot{W}(0) = 0$. Eq. (I-16) was used to compute the bubble wall motion displayed in fig. I-1.

To complete this section let us transform eq. (I-12) into the equilibrium system. We obtain ($\tau_y = t/[R_e(\varrho/p_\infty)^{1/2}]$, $Y = R/R_e$)

$$\ddot{Y}Y + \tfrac{3}{2}\dot{Y}^2 = Y^{-3\gamma} - 1\,. \tag{I-17}$$

Now the initial conditions are $Y(0) = Y_M = A$ and $\dot{Y}(0) = 0$.

4. PRESSURE AND VELOCITY FIELDS

Let us find the pressure field $p(r, t)$ in the liquid surrounding an oscillating bubble. This pressure field can be determined from an equation [1]:

$$\frac{1}{\varrho}\frac{\partial p}{\partial r} = -\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial r}\,. \tag{I-18}$$

The partial derivatives $\partial v/\partial t$ and $\partial v/\partial r$ can be obtained from (I-3). Then, upon integrating (I-18) from r to ∞ and from p to p_∞, we obtain

$$\left(\frac{p}{p_\infty} - 1\right)\frac{p_\infty}{\varrho} = \frac{R}{r}(\ddot{R}R + 2\dot{R}^2) - \left(\frac{R}{r}\right)^4 \tfrac{1}{2}\dot{R}^2\,. \tag{I-19}$$

This equation gives the pressure p as a function of time t and a position r. For a given point r eq. (I-19) gives a time dependence of the pressure in that point (fig. I-2) and for a given t eq. (I-19) gives a space distribution of the pressure in the liquid (fig. I-3).

Let us briefly analyse the pressure eq. (I-19). The acoustic pressure in the liquid is defined as $p_a = p - p_\infty$ [27]. Then the peak and valley acoustic pressures are $p_p = p_M - p_\infty$ and $p_v = p_m - p_\infty$, respectively (fig. I-2). In a similar way the acoustic pressure in the gas (at the bubble wall) can be defined as $P_a = P - p_\infty$. Hence, the equations of motion (I-12), (I-16) or (I-17) can be rewritten in the dimensional form

$$\ddot{R}R + \tfrac{3}{2}\dot{R}^2 = \frac{P_a}{\varrho}\,. \tag{I-20}$$

Substituting (I-20) into (I-19) gives

$$p_a = \frac{R}{r} P_a + \frac{R}{r} \tfrac{1}{2}\varrho\dot{R}^2 - \left(\frac{R}{r}\right)^4 \tfrac{1}{2}\varrho\dot{R}^2 . \tag{I-21}$$

Eq. (I-21) has three terms on its right-hand side. The first term, $(R/r)\,P_a$, represents a pressure communicated from the gas into the liquid. It is determined solely by the acoustic pressure in the gas and, as will be shown later [28], it is also the only component from the right-hand side of eq. (I-21) that remains present when the bubble oscillates linearly.

The second term, $(R/r)\,\frac{1}{2}\varrho\dot{R}^2$, is due to the dynamic motion of the bubble wall. It is significant only for nonlinear bubble oscillations. The first and the second term change as $1/r$ with distance and therefore they represent a prevailing pressure at sufficiently large distances. Since they change as $1/r$, which is typical of spherical acoustic waves, these two terms will be referred to as the acoustic field.

The third term, $(R/r)^4\,\frac{1}{2}\varrho\dot{R}^2$, is also due to the dynamic motion of the bubble wall and is therefore significant only for nonlinear bubble oscillations. Since it changes as r^{-4} with distance it is important only at points in the liquid adjacent to the bubble wall and at larger distances it can be neglected. Since from the equation of continuity for noncompressible liquid (I-3) it follows that

$$\left(\frac{R}{r}\right)^4 \tfrac{1}{2}\varrho\dot{R}^2 = \tfrac{1}{2}\varrho v^2 , \tag{I-22}$$

this term is also called the Bernoulli pressure or the kinetic wave [2]. Here this term will be referred to as the kinetic field.

The acoustic and kinetic fields are parts of the pressure field. On the other hand, the near and far fields are parts of the velocity field [27]. The near field changes as r^{-2} with distance and is reactive, i.e. it does not transfer acoustic energy. The far field changes as $1/r$. From the equation of the velocity field (I-3) it follows that only the near field is present in Rayleigh's model.

To characterize the bubble pulse form we shall define an effective pulse width, ϑ, as

$$\vartheta = \frac{1}{p_p^2} \int_{t_1}^{t_2} p_a^2 \, dt , \tag{I-23}$$

where t_1 and t_2 correspond to two subsequent moments when the valley pressure p_v occurs in the wave. Thus defined ϑ represents the width of a rectangular pulse that would carry away the same acoustic energy as the bubble pulse.

For further work it will be useful to transform. eq. (I-19) into the nondimensional form. To determine the space distribution of the pressure in the liquid the following equation will be used ($p^* = p/p_\infty$, $z = r/R_M$)

$$p^* = 1 + Z(\ddot{Z}Z + 2\dot{Z}^2)\,z^{-1} - \tfrac{1}{2}Z^4\dot{Z}^2 z^{-4} . \tag{I-24}$$

This equation can also be used to determine the time dependence of the pressure in a point z (the waveform). However, in finding the waveform we shall limit ourselves only to the acoustic field, which represents the most important case from the experimental point of view. Then the kinetic field can be omitted and (I-24) rearranged to give

$$p_z = Z(\ddot{Z}Z + 2\dot{Z}^2)\,, \quad z \gg 1\,, \tag{I-25}$$

where the position independent acoustic pressure is defined as $p_z = (p/p_\infty - 1) \cdot (r/R_M)$.

After substitution of eqs. (I-11) and (I-13) into (I-24), one obtains an algebraic equation in Z and z. Taking Z as a parameter, p^* can be easily determined as a function of z. An example of the pressure distribution in the liquid for the case of nonlinear bubble oscillations is given in fig. I-3.

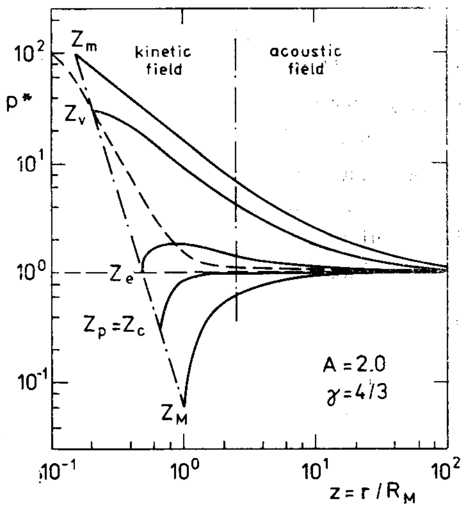

Fig. I-3. Distribution of the pressure in the liquid.

It follows from (I-24) that the pressure in the wave varies as $1/r$ only at larger distances z (acoustic field). In the vicinity of the nonlinearly oscillating bubble the inverse law $1/r$ holds only for the peak value p_p and the valley value p_v. Owing to the presence of the kinetic field the equilibrium pressure, P_e, at the bubble wall corresponds to the pressure in the liquid, p, which exceeds the pressure at infinity, i.e. $p > p_\infty$. Let us denote the wall position for which $p = p_\infty$ at the acoustic field (and hence $p_z = 0$) as Z_p.

It may be seen in fig. I-3 that for a given value of Z the pressure field in the liquid has a maximum p^*_{max}, at a certain point. This point, z_{max}, may be found from eq. (I-24). We obtain

$$z_{max} = Z\left(\frac{2\dot{Z}^2}{\ddot{Z}Z + 2\dot{Z}^2}\right)^{1/3}. \tag{I-26}$$

Using eqs. (I-24) and (I-26) the corresponding values of z_{max} and p^*_{max} can be found. The locus of these maxima is displayed in fig. I-3 by the dashed line.

Let us analyse eq. (I-26). The wall is supposed to move from Z_M to Z_m (the compression phase). It is evident that only the values of $z_{max} \in \{Z, \infty)$ have a physical meaning.

For $Z \in \langle Z_M, Z_p)$ the value of expression $(\ddot{Z}Z + 2\dot{Z}^2)$ is negative. Hence eq. (I-26) gives $z_{max} < 0$, which, as said above, has no physical meaning. However, from (I-24) it follows that the admissible pressure maximum is situated at $+\infty$. If the bubble wall moves further behind Z_p, the pressure maximum begins moving from $+\infty$ toward the bubble wall. The instant the wall velocity assumes a maximum ($\ddot{Z} = 0$, $\dot{Z} = \dot{Z}_{max}$, $Z = Z_v$), the pressure maximum enters the wall ($z_{max} = Z = Z_v$). For later times, i.e. when $Z \in \langle Z_m, Z_v)$, according to eq. (I-26), the pressure maximum occurs in the bubble interior. This again has no physical meaning. However, from eq. (I-24) or from fig. I-3 it follows that the admissible pressure maximum occurs at the wall. Nevertheless, it is interesting to follow the movement of the maximum formally even in the bubble interior. We find that as $Z \to Z_m$, then $z_{max} \to 0$, and $p^*_{max} \to \infty$. Hence, the movement of the maximum formally resembles the behaviour of a converging spherical wave. At $Z = Z_m$ the wall motion reverts and the pressure maximum travels just in the opposite order from the bubble centre back to $+\infty$. However, it should be stressed here that the velocity with which the pressure maximum travels is determined solely by the wall velocity and has nothing in common with the sound speed in the liquid, which is infinite in Rayleigh's model.

The velocity field (I-3) has a nondimensional form

$$\dot{z} = \dot{Z}Z^2 z^{-2} . \tag{I-27}$$

Just as in the case of the pressure field, it is often convenient to work with a position independent velocity field, v_z, defined as $v_z = \dot{z}z^{-2}$. Then from (I-27) it follows that

$$v_z = \dot{Z}Z^2 . \tag{I-28}$$

The velocity field equals zero for $Z = Z_M = 1$ and $Z = Z_m$. It has a maximum, $v_{z\max}$, when $Z = Z_c$. This maximum occurs at the same moment when $p_z = 0$ (as said above, the velocity field is a reactive near field). Hence, the wall positions Z_c and Z_p coincide, i.e. $Z_c = Z_p$. It will be shown later [28] that in the case of linear bubble oscillations the wall velocity and particle velocity maxima occur at the same moment, i.e. it holds that $X_v = X_c = X_p = X_e$ $(X = (R - R_e)/R_e)$. However, in the case of nonlinear oscillations the wall velocity and the velocity field attain maxima at different moments, i.e. $Z_v \neq Z_c$.

In conclusion of this section let us give a few remarks. First, the nondimensional form of the effective bubble pulse width in the acoustic field is given by

$$\vartheta_z = \frac{1}{p_{zp}^2} \int_{\tau_{z1}}^{\tau_{z2}} p_z^2 \, d\tau_z , \quad z \gg 1 , \tag{I-29}$$

where $\vartheta_z = \vartheta/[R_M(\varrho/p_\infty)^{1/2}]$. Further, since $Z(0) = 1$ and $\dot{Z}(0) = 0$ it follows from (I-25) that $p_{zv} = P_m^* - 1$. Similarly it follows for the peak pressure that $p_{zp} = (P_M^* - 1) Z_m$.

5. SIGNIFICANT WALL POSITIONS

As mentioned in the previous sections there are several significant wall positions. These are, for example, positions for which the variables $\dot{Z}$, $\ddot{Z}$, p_z and $\dot{v}_z$ equal zero. In the case of the wall velocity, $\dot{Z}$, this happens when $Z = Z_M$ and $Z = Z_m$. The wall acceleration, $\ddot{Z}$, equals zero for $Z = Z_v$, the pressure, p_z, equals zero (in the acoustic field) for $Z = Z_p$, and the particle acceleration, $\dot{v}_z$, is zero for $Z = Z_c$. In the Z system it holds that $Z_M = 1$. The values of Z_m and Z_v can be obtained from eqs. (I-11) and (I-13), respectively, and Z_p can be determined from (I-25) after substituting eqs. (I-11) and (I-13) for $\dot{Z}$ and $\ddot{Z}$. Finally, Z_c can be found as a maximum of eq. (I-28). Z_e can be considered to be another significant position whose value is $Z_e = 1/A$ by definition. Let us note here that besides the property $P_e = p_\infty$ when $Z = Z_e$ the equilibrium position can also be characterized by the fact that the kinetic energy of the liquid attains a maximum, $E_{zk\,max}$, when $Z = Z_e$.

The meanings and some interesting features of the significant wall positions are summarized in table I. Their dependence on amplitude A is displayed in fig. I-4.

Table I

Property and determination of significant wall positions.

Position	Property	Determination of the position		Limit for $A \to 1$	Limit for $A \to \infty$
Z_M	$\dot{Z} = 0$	$Z_M = 1$		$Z_M = 1$	$Z_M = 1$
$Z_p = Z_c$	$p_z = 0$ $v_{z\,max}$	$Z_p = Z_c = (\frac{1}{4} + \frac{3}{4}A^{-4})^{1/3}$, Newton's method	$\gamma = \frac{4}{3}$ $\gamma \neq \frac{4}{3}$	$Z_p = Z_c = 1$	$Z_p = Z_c = 0{\cdot}63$
Z_e	$P_e^* = 1$ $E_{zk\,max}$	$Z_e = 1/A$		$Z_e = 1$	$Z_e = 0$
Z_v	$\ddot{Z} = 0$ $\dot{Z}_{max}$	$Z_v = \left(\frac{\gamma}{1 + (\gamma - 1) \cdot A^{3\gamma}}\right)^{1/[3(\gamma-1)]}$		$Z_v = 1$	$Z_v = 0$
Z_m	$\dot{Z} = 0$	Cardan's formula Newton's method	$\gamma = \frac{4}{3}$ $\gamma \neq \frac{4}{3}$	$Z_m = 1$	$Z_m = 0$

$$Z_M \geqq Z_p = Z_c \geqq Z_e \geqq Z_v \geqq Z_m$$

To demonstrate some advantages of the Y system (here, for example, a symmetry), the equilibrium system was used instead of the compression system in this figure. As $Y = Z \cdot A$, the conversion between the two systems is very simple.

In table I the determination of the significant positions for given A and γ is also mentioned. Let us have a closer look at the determination of the position Z_m. This position is rather important as it is used to calculate P_M^* and p_{zp}. The position Z_m can be found by solving eq. (I-11), which can be done in two ways. First, for given Z_m and γ an explicit expression for A can be found. Thus it is possible to obtain exact values of A, γ, and Z_m. On the other hand, for given A and γ the position Z_m can be found numerically, e.g. by Newton's method. The case when $\gamma = 4/3$ is especially

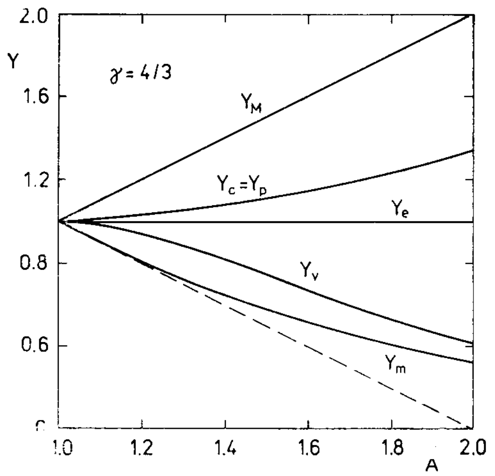

Fig. I-4. Dependence of the significant wall positions on amplitude A.

simple. Then (I-11) represents an algebraic equation of the fourth degree. As one root is known ($Z_M = 1$), the degree of this equation can be reduced by one. The resulting equation is of the third degree and can be further solved by Cardan's formula.

The procedures for finding the significant positions will be used later [28] to determine scaling and independent functions. The knowledge of the exact values Z_v and Z_m also proved to be useful in checking the results of numerical integrations performed by the Runge-Kutta method. Based on comparison of errors, an efficient computation scheme, which saved much of the machine time, could be developed.

6. DISCUSSION

In this paper after derivation of the equation of motion the main attention was paid to the examination of the pressure and velocity fields. The reason for it is that, as far as the author is aware, no detailed study of this sort has been published yet, though radiated waves represent a very important source of information on bubble behaviour. However, it should be stressed here that, whereas interpretation of the bubble dynamics in Rayleigh's model presents no serious problem, interpretation of the acoustic radiation requires a certain amount of caution. This is so because the sound speed in an incompressible liquid is infinite and therefore a disturbance is communicated throughout the whole volume of the liquid instantly, i.e. the wave occurs at a point r at the same moment t it was radiated by the bubble. Another consequence of the infinite sound speed is the nonexistence of the far field. Hence, only a reactive near field is present in Rayleigh's model and the radiated waves do not transfer acoustic energy. Therefore there is no radiation damping either. Nor does the considered model account for the propagation of the shock waves and further nonlinear acoustic effects.

Though the equations derived are valid quite generally, the excitation of gas bubbles by immediate heat supply was preferred in section 2. Such an excitation occurs, for example, in the case of underwater explosions [2], electric discharges [29, 30] and laser generated explosions [23, 31, 32]. As processes involved in these real situations are highly irreversible and nonlinear, the simple Rayleigh's model represents only a very rough approximation to them, especially in the vicinity of the minimum radius. Nevertheless, in view of still insufficient knowledge of the phenomena associated with the explosive excitation and intensive oscillations of bubbles the amount of information that can be obtained using this model is even so considerable.

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