TIME-FREQUENCY STATISTICAL CHARACTERISTICS OF CYCLOSTATIONARY SIGNALS

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ABSTRACT

Time-frequency statistical characteristics of cyclostationary signals are studied. The characteristics considered are real and complex instantaneous autospectra. Unfortunately, as both these characteristics can take on negative values, they are not spectral densities. Moreover, the second characteristics is a complex function of time and frequency. The first characteristics, on the other hand, can be deteriorated by artifacts occurrence. The origin of these artifacts is shown to be the difference between the support area, on which the autospectrum is unambiguously defined, and the area, which represents the basic period of the autospectrum. However, both characteristics can be computed with very good time and frequency resolution.

1. INTRODUCTION

A number of phenomena in nature, society, and technology have cyclic character, i.e. they repeat to some extent more or less periodically. Rotation of the Earth on one hand and working cycles of engines, turbines, and compressors on the other hand are just a few examples. Signals acquired by observers of these and associated phenomena have statistical characteristics which are periodic functions of time. Hence such signals are called cyclostationary [1], periodically correlated [2], or periodically nonstationary [2]. Traditional approach used to analyze these signals is based on assumption of their stationarity. This assumption implies continuous time averaging. Statistical characteristics obtained in this way are well known autospectral densities and autocorrelation functions. However, during continuous time averaging a certain amount of information, concerning, e.g., phase magnitude or time-frequency behavior of signals, is lost. Therefore methods have been looked for to preserve this information. They all are based on periodic time averaging and the resulting statistical characteristics are known as periodic mean [1], gated autospectrum [3], cyclic autocorrelation and cyclic autospectrum [1], spectral correlation [4], double autocorrelation and double autospectrum [5], etc. In the paper some statistical characteristics that are related to time-frequency behavior of cyclostationary signals will be discussed.

At the first sight the approach used in time-frequency analysis of cyclostationary signals may seem to be the same as that used for transient signals. However, as will be shown, definite and important differences exist. These differences follow from the periodic nature of cyclostationary signal statistical characteristics.

Motivation for the research described in this paper has been an endeavor to improve analysis of noise and vibration generated by reciprocating machinery. Results obtained by other researchers in this field can be found, e.g., in references [6-12].

2. STATISTICAL CHARACTERISTICS OF CYCLOSTATIONARY SIGNALS

Time-frequency statistical characteristics of cyclostationary signals can be obtained both from time and frequency domain characteristics. In the following an approach based on time domain autocorrelation functions will be used. However, the same results would also be obtained via frequency domain characteristics.

A double autocorrelation function is defined as [5]

$$R_{xx}(t_1,t_2) = E\left[x(t_1)x(t_2)\right] \quad . \tag{1}$$

Here $E[\ ]$ denotes averaging operation on the ensemble of time signals $x(t)$. A different autocorrelation function can be derived from (1) by introducing new variables

$$t = (t_1 + t_2)/2 \,, \quad \tau = t_2 - t_1 \quad . \tag{2}$$

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After substituting (2) into (1) one obtains [5]

$$R_{xx}(\tau,t)=E\left[x(t-\tau/2)x(t+\tau/2)\right] . \quad (3)$$

In the case of cyclostationary signals the autocorrelation functions $R_{xx}(t_1,t_2)$ and $R_{xx}(\tau,t)$ are computed by periodic averaging over different periods of signal $x(t)$ and both $R_{xx}(t_1,t_2)$ and $R_{xx}(\tau,t)$ are periodic functions of the respective variables. In the case of $R_{xx}(t_1,t_2)$ it follows from the definition (1) that

$$R_{xx}(t_1,t_2)=R_{xx}\left(t_1+nT_p,t_2+mT_p\right) , \quad (4)$$

where T_p is the period of cyclostationarity and n, $m=0$, ± 1, ± 2, ... Hence the basic period of $R_{xx}(t_1,t_2)$ in the plane (t_1,t_2) is given by an area

$$t_1 \in \langle -T_p/2,T_p/2\rangle,\ \ t_2 \in \langle -T_p/2,T_p/2\rangle . \quad (5)$$

As far as $R_{xx}(\tau,t)$ is concerned, its periodicity is

$$R_{xx}(\tau,t)=R_{xx}\left(\tau+2nT_p,t+mT_p\right) . \quad (6)$$

Hence the basic period of $R_{xx}(\tau,t)$ in the plane (t,τ) is given by an extended area

$$t \in \langle -T_p/2,T_p/2\rangle,\ \ \tau \in \langle -T_p,T_p\rangle . \quad (7)$$

But the autocorrelation function $R_{xx}(\tau,t)$ is unambiguously defined on the support area

$$t \in \langle -T_p/2,T_p/2\rangle,\ \tau \in \langle -T_p+2|t|,T_p-2|t|\rangle . \quad (8)$$

By comparison of (7) and (8) it can be seen that the support area has a half size of the extended basic area.

The time-frequency statistical characteristics of cyclostationary signals $x(t)$ can be obtained by a direct Fourier transform of the two autocorrelation functions. In the first case one obtains a complex instantaneous autospectrum

$$W_{xx}(f_1,t_2)=\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} R_{xx}(t_1,t_2)e^{-j2\pi f_1 t_2}dt_2 . \quad (9)$$

The complex instantaneous autospectrum $W_{xx}(f_1,t_2)$ is a periodic function of time

$$W_{xx}(f_1,t_2)=W_{xx}\left(f_1,t_2+mT_p\right) , \quad (10)$$

and a complex conjugate function of frequency

$$W_{xx}(f_1,t_2)=W_{xx}^{*}(-f_1,t_2) . \quad (11)$$

Real and imaginary parts of $W_{xx}(f_1,t_2)$ can take on both positive and negative values.

In the second case one obtains a real instantaneous autospectrum (cyclic Wigner distribution)

$$W_{xx}(f,t)=\int_{-T_p}^{T_p} R_{xx}(\tau,t)e^{-j2\pi f\tau}d\tau . \quad (12)$$

The real instantaneous autospectrum $W_{xx}(f,t)$ is a periodic function of time again, i.e.

$$W_{xx}(f,t)=W_{xx}\left(f,t+mT_p\right) , \quad (13)$$

and a real-valued and even function of frequency, i.e.

$$W_{xx}(f,t)=W_{xx}^{*}(f,t),\ W_{xx}(f,t)=W_{xx}(-f,t). \quad (14)$$

Unfortunately it also can take on negative values. Because the extended basic period area is not coincident with the support area, the function $W_{xx}(f,t)$ also contains artifacts that deteriorate its interpretation.

As both instantaneous autospectra can take on negative values, they are not spectral densities. Nevertheless their information content is rich. First, they show time-frequency behavior of cyclostationary signals. Second, they contain information concerning the phase magnitude of cyclostationary signals. If for some reason time-frequency distribution of energy in signal (time-frequency autospectral density) is required, gated autospectrum [3] can be used to show it. Unfortunately, time and frequency resolutions that can be obtained with gated autospectrum are much worse than with the instantaneous autospectra [3].

3. SIMULATED SIGNAL

To show the properties of the studied statistical characteristics, a suitable cyclostationary signal $x(t)$ was simulated on a computer and then analyzed using both the complex and real instantaneous autospectra $W_{xx}(f_1,t_2)$ and $W_{xx}(f,t)$, respectively. The signal was a train of 100 square pulses with a random amplitude (mean value $\mu_x=0$ V, standard deviation $\sigma_x=1$ V). This random form of the signal was selected purposefully to show ability of the methods studied here to reveal the pulse form even in those cases, where techniques, such as synchronous averaging, would fail. The pulses occurred periodically with a period $T_p=1$ ms, their width was 62.5 μs and they were situated in the middle of each period. The sampling frequency was 128 kHz. The first 6 periods of the simulated random pulse train $x(t)$ are shown in Fig. 1.

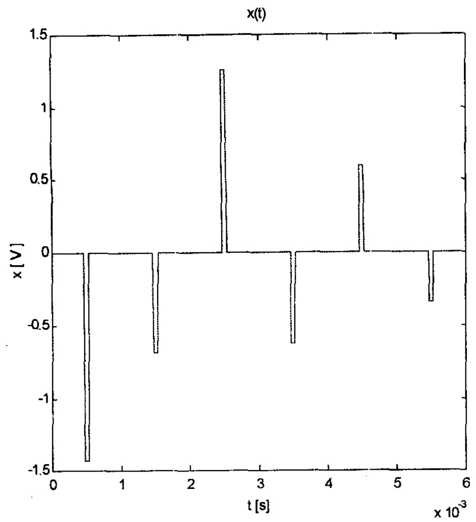

Fig. 1.: Simulated random pulse train signal $x(t)$.

The double autocorrelation functions $R_{xx}(t_1,t_2)$ and $R_{xx}(\tau,t)$ were computed by averaging over 100 periods and then Fourier transformed to obtain the complex and real instantaneous autospectra $W_{xx}(f_1,t_2)$ and $W_{xx}(f,t)$, respectively. An example of the computed instantaneous autospectrum $W_{xx}(f_1,t_2)$ is displayed in Fig. 2. From Fig. 2 it follows that $W_{xx}(f_1,t_2)$ shows the time-frequency behavior of the signal $x(t)$ within the basic period without any distortion and artifacts.

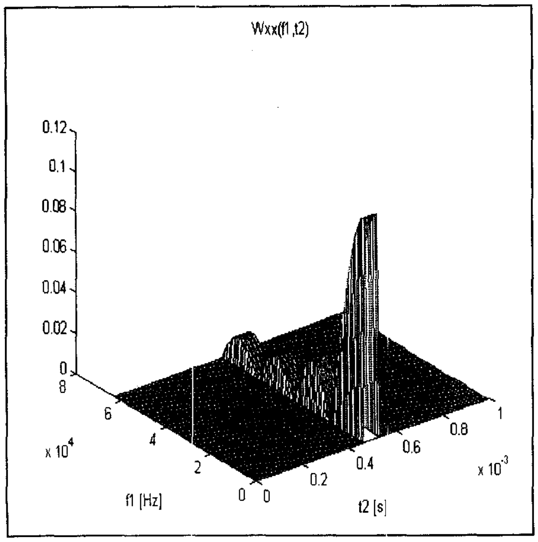

Fig. 2. Magnitude of the complex instantaneous autospectrum $W_{xx}(f_1,t_2)$ of a pulse train $x(t)$.

An example of the computed real instantaneous autospectrum $W_{xx}(f,t)$ is displayed in Fig. 3. Even if this is not necessary with real autospectrum, for the purpose of a better comparison, magnitude of $W_{xx}(f,t)$ is shown. Two artifacts, one at the beginning and the other at the end of the basic period can be seen in Fig. 3. The artifacts distort the spectrum significantly and it is evident that with more complex signals this distortion could be a serious problem.

4. REAL SIGNALS

Real signals, such as vibration and noise measured on rotating machinery, are seldom strictly cyclostationary. Hence their conditioning prior to further processing is usually required. One possible way to solve this problem is a parallel recording of the signal $x(t)$ and tachopulses. Tachopulses are used to define the period of cyclostationarity and the signal is then resampled to have the same number of samples in each basic period T_p. However, other techniques, such as adaptive processing can be used as well.

High complexity of real signals may represent another problem. To resolve different discrete spectral components, spectral resolution must usually be very high over a wide range of frequencies. If this is the case, then computation of $W_{xx}(f_1,t_2)$ and $W_{xx}(f,t)$ within the whole basic period would be impractical. However, closer examination of formulas (1), (3), (9) and (12) reveals that it is possible to limit computations just to the region of

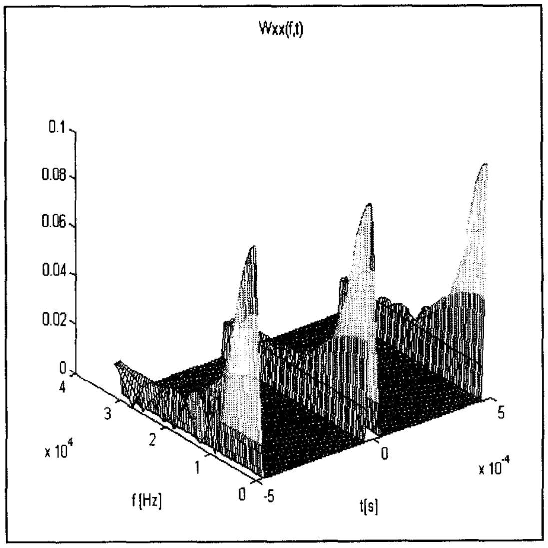

Fig. 3. Magnitude of the real instantaneous autospectrum $W_{xx}(f,t)$ of a pulse train $x(t)$.

interest and thus lower computational burden to an acceptable value. Yet another problem may be the length of signals needed to compute statistical characteristics with a high spectral resolution, because at present time equivalents of techniques, like overlapped processing used in standard spectral methods, are not known for cyclostationary signals.

5. CONCLUSION

In the paper time-frequency statistical characteristics of cyclostationary signals were studied. The characteristics considered were the complex and real instantaneous autospectra $W_{xx}(f_1,t_2)$ and $W_{xx}(f,t)$, respectively. Unfortunately, both these characteristics can take on negative values and therefore they are not spectral densities. Even more, the first characteristics is a complex function of time and frequency. On the other hand, the real instantaneous autospectrum can be deteriorated by artifacts occurrence. It was shown that the origin of these artifacts is the difference between the support area, on which the real autospectrum is unambiguously defined, and the area, which represents the basic period of the real autospectrum. However, it should be stressed here that both these characteristics can be computed with excellent spectral and time resolutions. When necessary, gated autospectrum can be used to obtain time-frequency autospectral density of cyclostationary signals relatively easily. Unfortunately, time and frequency resolutions that can be obtained with this technique are very low [3].

Methods discussed in this paper have been developed for applications encompassing acoustic noise reduction, acoustic quality control and predictive maintenance of rotating machinery. Real signals, such as vibration and noise measured on rotating machinery usually need conditioning to make them cyclostationary. This may require recording suitable tachopulses for definition of the basic period and subsequent signal resampling. Further research is needed to find effective methods for computing statistical characteristics with high time and frequency resolutions using reasonably short signal lengths and computation times.

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