A simple model of a vapor bubble

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A simple yet reasonably accurate vapor bubble model based on switching between an empty bubble and a gas bubble is introduced. The switching bubble wall velocity is determined by adjusting the theoretical model to experimental data. A comparison with gas bubble models is given.

INTRODUCTION

By definition, a pure *gas bubble* is assumed to contain in its interior only a noncondensable gas.[1,2] The theory of gas bubbles is rather well elaborated and there are numerous publications covering this subject (see, e.g., Refs. 3 and 4).

A pure *vapor bubble*, on the other hand, is assumed to contain only the vapor of the surrounding liquid.[1,2] Furthermore, it is assumed that, at some stages of the vapor bubble life, condensation or evaporation takes place. Over the years a number of vapor bubble models have been described in the literature. These models differ primarily in assumptions about the speed at which the processes of condensation and evaporation are assumed to take place. The simplest vapor bubble model assumes that condensation and evaporation can take place at an infinite speed.[5] In such a model, which is practically identical with Rayleigh's model of an empty bubble,[6] the bubble grows to a maximum radius and then collapses. At the final stages of the collapse, when the bubble wall approaches the bubble center, the wall velocity and acceleration grow to infinity.[6] Although this model is evidently unrealistic, its simplicity makes it rather attractive.

A somewhat better approximation is based on the assumption that the speed of condensation and evaporation is zero. In this case the vapor bubble is modeled by a gas bubble. Because of its simplicity, this model is rather often used,[7–9] but its limitations are obvious.

Apart from the two approaches mentioned, there are a number of works which endeavor to model the vapor bubbles as realistically as possible.[10–13] Unfortunately, these models are rather complicated and hence time demanding.

The objective of this paper is to present a model which, on one hand, is more realistic than the oversimplified empty and gas bubble models, but which, on the other hand, retains their simplicity. To keep the exposition as simple as possible, we will consider only medium-sized spherical bubbles, for which the effect of gravity, surface tension, viscosity, and heat conduction can be neglected.[6,14]

I. THE EQUATION OF MOTION

The formulation of the bubble model can be made in two steps. First, it is necessary to find an approximation for the bubble content behavior. This will be done in Sec. II. Second,

the liquid behavior must be described in a suitable way. Such a description results in the equation of motion for the bubble wall and in equations of the pressure and velocity fields. The equation of motion can be derived from governing hydrodynamical relations, and the bubble content behavior enters this equation only through the boundary value.

A number of equations of motion have been given in the literature (see, e.g., Refs. 4 and 15). These equations differ, first, with regard to the accuracy with which the liquid compressibility is approximated. In this paper we shall work with Herring's modified equation, which has the form[15]

$$\ddot{R}R + \tfrac{3}{2}\dot{R}^2 = (P - p_\infty + \dot{P}R/c_\infty)/\rho_\infty. \qquad (1)$$

Here, R is the bubble radius, ρ_∞ is the liquid density, P is the pressure in the liquid at the bubble wall, p_∞ is the ambient pressure in the liquid, and c_∞ is the velocity of sound in the liquid. The overdots denote differentiation with respect to time.

Equation (1), in spite of its relative simplicity, performs rather well and gives satisfactory results for a broad range of bubble oscillation intensities.[15] As will be shown later (cf. Table II), the vapor bubbles oscillate with intensities for which the use of Eq. (1) is fully justified.

An oscillating bubble radiates pressure waves into the surrounding liquid. If the propagation of these waves can be treated in the frame of linear acoustics, then the peak pressure in the bubble pulse p_p equals[6]

$$p_p = p_M - p_\infty = (P_M - p_\infty)R_m/r. \qquad (2)$$

Here, p_M is the maximum pressure in the bubble pulse at a point r, R_m is the minimum bubble radius, and P_M is the (maximum) pressure at the bubble wall when $R = R_m$.

II. BUBBLE COLLAPSE

Let us assume that a bubble has been excited and begins to grow.[16] After reaching a maximum radius R_M, the bubble wall starts moving inwards. In the case of vapor bubbles this inward motion is usually referred to as the bubble collapse.[1]

If the bubble collapse is modeled by an *empty bubble*, it is then assumed that the pressure and temperature at the bubble wall remain constant; i.e., one assumes[5]

$$P = P_v, \quad \theta = \theta_\infty, \qquad (3)$$

where P_v is the liquid vapor pressure and θ_∞ is the liquid temperature.

If the vapor bubble behavior is modeled by a *gas bubble*,

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the equations for the pressure and temperature at the bubble wall have the form, respectively,[7-9]

$$P = P_m (R_M/R)^{3\gamma} , \tag{4}$$

$$\theta = \theta_\infty (R_M/R)^{3(\gamma-1)} . \tag{5}$$

Here, P_m is the minimum pressure (i.e., when $R = R_M$) at the bubble wall and γ is the polytropic exponent of the vapor.

Since in real *vapor bubbles* the rate at which the process of condensation takes place is always finite and nonzero, it is expedient to divide the bubble wall velocities and positions into two regions. If the wall velocity $\dot{R}$ in the vicinity of the maximum radius R_M is lower than a certain velocity $\dot{R}_{vg}$, we can assume that the process of condensation takes place at such a speed as to maintain the pressure in the bubble interior P equal to the vapor liquid pressure P_v and the vapor temperature θ equal to the liquid temperature θ_∞. Thus, in this low-velocity region, which is defined by the condition that $|\dot{R}| < |\dot{R}_{vg}|$ and simultaneously $R > R_{vg}$, we use Eqs. (3).

On the other hand, if the wall velocity $\dot{R}$ exceeds the velocity $\dot{R}_{vg}$, we shall assume that there will be no condensation at all, which means that the bubble starts behaving like a gas bubble. Thus, if $|\dot{R}| \geqslant |\dot{R}_{vg}|$ and $R < R_{vg}$, we have

$$P = P_v (R_{vg}/R)^{3\gamma} , \tag{6}$$

$$\theta = \theta_\infty (R_{vg}/R)^{3(\gamma-1)} . \tag{7}$$

Let us note that this conditional switching between Eqs. (3) and (6) and (7) can be easily programmed on a digital computer. Also, the respective equations were formulated independently of any assumption on the liquid compressibility, and thus they can be used in conjunction with any equation of motion for the bubble wall.

III. NUMERICAL RESULTS

For further work, it is convenient to introduce nondimensional variables. In this paper, the nondimensional radius, time, pressure, and temperature at the bubble wall, and the peak pressure in the bubble pulse, respectively, are defined as[6]

$$Z = \frac{R}{R_M}, \quad t_z = \frac{t}{[R_M(\rho_\infty/p_\infty)^{1/2}]}, \quad P^* = \frac{P}{p_\infty},$$

$$\theta^* = \frac{\theta}{\theta_\infty}, \quad p_{zp} = \frac{(p_p/p_\infty)r}{R_M} .$$

Using these nondimensional variables, Eqs. (1)–(7) now take the form

$$\ddot{Z}Z + \tfrac{3}{2}\dot{Z}^2 = P^* - 1 + \dot{P}^*Z/c_\infty^* , \tag{8}$$

$$p_{zp} = (P_M^* - 1)Z_m , \tag{9}$$

where, if $|\dot{Z}| < |\dot{Z}_{vg}|$ and $Z > Z_{vg}$, the pressure P^* equals

$$P^* = P_v^* . \tag{10}$$

Besides, it holds that

$$\theta^* = 1 . \tag{11}$$

If $|\dot{Z}| \geqslant \dot{Z}_{vg}|$ and $Z < Z_{vg}$, the pressure P^* is given by the relation

$$P^* = P_v^*(Z_{vg}/Z)^{3\gamma} , \tag{12}$$

TABLE I. Computed values of significant bubble wall positions.

	$\gamma = 1.25$			$\gamma = 1.33$		
P_m^*	0.02	0.02	0.012	0.02	0.02	0.0067
$\dot{Z}_{vg}$	-0.6	$\cdots$	$\cdots$	-0.91	$\cdots$	$\cdots$
Z_{vg}	0.864	$\cdots$	$\cdots$	0.761	$\cdots$	$\cdots$
Z_e	0.304	0.352	0.304	0.286	0.376	0.286
Z_m	0.0307	0.0464	0.0306	0.0343	0.0706	0.0342

and the temperature equals

$$\theta^* = (Z_{vg}/Z)^{3(\gamma-1)} . \tag{13}$$

Finally, in the case of the gas bubble, we have

$$P^* = P_m^* Z^{-3\gamma} \tag{14}$$

and

$$\theta^* = Z^{-3(\gamma-1)} . \tag{15}$$

The nondimensional velocity c_∞^* in Eq. (8) is defined as $c_\infty^* = c_\infty(\rho_\infty/p_\infty)^{1/2}$.

To determine the velocity $\dot{Z}_{vg}$ and the corresponding position Z_{vg}, a number of trial computations with Eqs. (8)–(10) and (12) were performed. In these computations the value of $\dot{Z}_{vg}$ was varied and the computed peak pressure in the bubble pulse compared with the experimental value $p_{zp} = 167$, as measured by Mellen[17] (see Ref. 18 for further details).

The computations were performed with the following values of physical constants (water under ordinary laboratory conditions):

$$p_\infty = 100 \text{ kPa}, \quad \rho_\infty = 10^3 \text{ kg m}^{-3}, \quad c_\infty = 1450 \text{ m s}^{-1},$$

$$P_v = 2 \text{ kPa}, \quad \theta_\infty = 293 \text{ K}, \quad \gamma = 1.25, \text{ and } \gamma = 1.33 .$$

The initial conditions of Eq. (8) were $Z(0) = 1$, $\dot{Z}(0) = 0$.

It was found that a satisfactory fit to the peak pressure $p_{zp} = 167$ is obtained when $\dot{Z}_{vg} = -0.6$ ($\gamma = 1.25$) and $\dot{Z}_{vg} = -0.91$ ($\gamma = 1.33$). The significant positions of the bubble wall computed for these values of $\dot{Z}_{vg}$ are given in the first and fourth columns of Table I. In Table I, Z_e is an equilibrium radius defined by the condition that $P^* = 1$ when $Z = Z_e$ (cf. Ref. 6). Also given in Table I are the values of the significant positions computed with the vapor bubble models based on the gas bubbles. In this case the pressure P^* is given by Eq. (14). These computations were performed both for P_m^* equal to the liquid vapor pressure P_v^* (the second and fifth columns in Table I) and for P_m^* determined from a condition that both the vapor and gas bubbles have the same amplitude of oscillations $A = R_M/R_e$ (the third and sixth columns in Table I). Note that the values of P_m^* given in Table I are rounded off in this case. The full values may be found from the relation $P_m^* = Z_e^{3\gamma}$ (cf. Ref. 16).

IV. COMPARISON OF MODELS

The values of significant positions given in Table I can be substituted into the definition equation for the amplitude of bubble oscillations $A = R_M/R_e = 1/Z_e$ and into Eqs. (9)

TABLE II. Computed values of significant bubble parameters.

	$\gamma = 1.25$			$\gamma = 1.33$		
P_m^*	0.02	0.02	0.012	0.02	0.02	0.0067
A	3.29	2.84	3.29	3.50	2.66	3.50
p_{zp}	167	93	168	166	57	167
P_M^*	5460	2000	5480	4830	800	4880
θ_M^*	12.2	10.0	13.7	22.2	14.2	29.2
θ_M (K)	3580	2930	4000	6500	4150	8560

and (12)–(15). The results obtained in this way are summarized in Table II.

When comparing the data presented in Tables I and II, we note several interesting facts. First, it can be observed that if the vapor bubble is modeled by the gas bubble and the pressure P_m^* is set equal to P_v^*, the resulting amplitude of the bubble oscillations is relatively small, so that the computed peak pressure p_{zp} is much lower than the measured pressure.

It is only because of the amplifying collapse effect that the vapor bubble attains a larger amplitude—and hence a higher maximum pressure and temperature—than the gas bubble having the same minimum pressure $P_m^* = P_v^*$.

If we model the vapor bubble by a gas bubble having the same amplitude A, we find that the pressures p_{zp}, P_M^* and the radii Z_e, Z_m are almost the same for the two models. The only larger difference lies in the values of the minimum pressure P_m^* and the maximum temperature θ_M^*.

However, it is in these values that the vapor bubble model provides more realistic results. Specifically, it should be stressed here that in no bubble can the pressure drop under the liquid vapor pressure at the given temperature. If the ambient pressure does drop under P_v, e.g., due to a tension wave, the liquid at the bubble wall will immediately start boiling[19] and the evaporation which thus results will maintain the minimum pressure in the bubble equal to P_v. Because of this, the values $P_m < P_v$ are physically unrealistic and, even if they are sometimes used for pure computational purposes, care must be taken when interpreting the results.

Let us also note that the maximum temperature θ_M in the vapor bubble is somewhat lower than in the gas bubble oscillating with the same amplitude. This is due to the temperature retardation associated with the collapse mechanism. However, it seems that even this lower temperature may be sufficient to produce sonoluminescence.[20]

V. CONCLUSION

The purpose of this paper was to introduce a simple, yet reasonably accurate, vapor bubble model. This model can easily be programmed on a digital computer and allows for fast and efficient computations. As we want to show elsewhere, the model may also help to explain several interesting facts connected with concrete excitation techniques where application of the gas bubble models leads to inappropriate results.

It was found that a satisfactory fit to the experimental peak pressure $p_{zp} = 167$ is obtained when $\dot{Z}_{vg} = -0.6$ ($\gamma = 1.25$) and $\dot{Z}_{vg} = -0.91$ ($\gamma = 1.33$). These wall veloc-

ities are in good agreement with the values determined in a different way by Plesset,[1] who found that $\dot{Z}_{vg} = -0.8$, and by Flynn,[21] who determined that $Z_{vg} = 0.85$ ($\dot{Z}_{vg} = -0.65$).

The isentropic exponent of the vapor γ is in no case constant for the range of pressures and temperatures considered.[22] Therefore, the constant γ used here should rather be regarded as a parameter affording a useful fit of simplified theory to experimental results. For this reason, the computations were performed with two values of γ which we felt were reasonably representative for the range considered. Evidently, further experimental data are necessary in order to determine which of these values is more suitable for the present problem.

In certain situations it may be convenient to model vapor bubbles by gas bubbles. This occurs, for example, when determining the amplitude of vapor bubble oscillations from experimental data by means of scaling functions computed with gas bubbles.[17] However, a great deal of care is necessary when considering the vapor bubble excitations because in this case the gas bubble model may yield highly erroneous results.

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